PHYS 328

Homework #1--Solutions

1. This question explores the relative cost of human vs. fossil fuel energy. A human being hard at work will expend approximately 100 W of power (if you use an exercise bike or treadmill that displays power, see how hard you have to work to expend 100 W). A small compact car traveling at highway speeds will expend approximately 100 kW. Use available data (such as the federal rate for reimbursing mileage) to determine how much money is required to operate a car for an hour. Now, suppose that the energy is provided by work study students pushing the car. How many students would you need to move the car at the same speed? Using the current U.S. minimum wage, how much would it cost to operate that car for an hour? Show all work and state explicitly all assumptions you make and values you use in your calculations.

Solution : The current federal mileage reimbursement rate is 56.5 cents/mile: (http://www.irs.gov/uac/2013-Standard-Mileage-Rates-Up-1-Cent-per-Mile-for-Business,-Medical-and-Moving) at highway speeds of say 55 mi/hr, the cost of operating a car for 1 hour is then 56.5 cents/mile x 55 mi = $31.08

Since the power generated by the internal combustion engine of the car is 1000 times greater than typical human power output, we would require the services of 1000 work study students to propel the car at the same speed. The current federal minimum wage is $7.25/hr, but the Illinois minimum wage is $8.25/hr. Using human power would cost $8250 in Illinois. Gas seems relatively cheap now, doesn't it?

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2. Determine the number of molecules in a cubic meter of air at the surface of the Earth at room temperature and 1 atm of pressure.

Solution : We begin with the perfect gas law:

$$\mathrm{PV} = \mathrm{N\,k\,T} \Rightarrow \mathrm{N} = \frac{\mathrm{P\,V}}{\mathrm{k\,T}} = \frac{10^5\ \mathrm{N\,m^{-2}} * 1\ \mathrm{m^3}}{1.38\,\mathrm{x}10^{-23}\ \mathrm{J/K} * 300\ \mathrm{K}} = 2.4 \times 10^{25}\ \mathrm{molecules}$$

The number of particles per unit volume at STP at the surface of the Earth is also known as *Loschmidt' s number*, and provides the basis for a unit called the *amagat*, which is a measure of particle number density (1 amagat is defined to equal the Loschmidt' s number).

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3. The number density of the atmosphere decreases exponentially with height according to :

$$\mathrm{n\,(z)} = \mathrm{n_0\,e^{-z/H}}$$

where n (z) is the number density at any height z above the surface of the Earth, n_o is the number density at the surface of the earth, and H is a constant called the scale height of the atmosphere. For the Earth, the scale height is 8 km.

Find the total number of molecules in a 1 m^2 column reaching from the surface of the Earth to the “top” of the atmosphere, several hundred kilometers above the surface of the Earth. Assuming the Earth is a sphere of radius R (R = 6400 km), estimate the total number of molecules in the Earth' s atmosphere.

Solution : We start by considering how many molecules are in a volume dV, given by:

$$dV = A\,dz$$

where A is the cross sectional area of the region and dz is the height of the region. We can set A to be a constant 1 m^2, so that the total number of molecules in this region is:

$$N(z) = n(z)\,dV = n(z)\,dz = n_0\,e^{-z/H}\,dz$$

Then, the total number of molecules in a column of this cross - sectional area is :

$$N_{total} = \int_0^{\text{top of atmosphere}} N(z)\,dz$$

Integrating simple exponential functions is easy, but what do we choose for our upper limit, since I did not give you a specific value for the "top of the atmosphere"? Let' s see if we can justify the use of infinity for our upper limit, because if we can, we get the particularly nice result :

$$N_{total} = H\,n_0$$

and we already know the value of n_0 from problem 4. How can we justify using ∞ as our upper limit since we know the atmosphere isn' t really infinite in extent? But we do know that the atmosphere extends to at least 80 km (which represents 10 scale heights), so we can estimate our fractional error in assuming an upper limit of infinity by calculating :

$$\int_{10}^{\infty} e^{-x}\,dx = e^{-10} = 4.5\times 10^{-5}$$

$$\left(\text{Some quick rules of thumb : } e^{-3} \approx \frac{1}{20};\ e^{-5} \approx \frac{1}{150}\right)$$

We can see that the error associated with using an upper limit of infinity is miniscule compared to other assumptions we have made (that I didn' t tell you about, mostly assuming an isothermal atmosphere to derive our exponential law) so that we can easily calculate that the total number of molecules in a column 1 m^2 in area is :

$$N_{total} = H\,n_0$$

and that the total number of molecules in the atmosphere is simply :

$$4\pi R^2\,H\,n_0 = 4\pi\left(6.4\times 10^6\,\text{m}\right)^2(8000\,\text{m})\left(2.4\times 10^{25}\,\text{molecules}/\text{m}^3\right) \approx 10^{44}\,\text{molecules}$$

4. ***Solutions***: a) We begin by writing the equation for the binomial distribution :

$$P(N, n) = \frac{N!}{n!\,(N-n)!}\, p^n q^{(N-n)}$$

The basic calculations will be performed by (showing explicitly the values for N = 100 and n = 50, notice that we cannot use N as a variable since that is a protected symbol in Mathematica) :

```
In[169]:= Clear[prob, p, q, ntotal, n]
prob[ntotal_, n_, p_, q_] := ntotal! / (n! (ntotal - n)!) p^n q^(ntotal - n)

prob[100, 50, 0.5, 0.5]

Out[171]= 0.0795892
```

The answer showing that the probability of obtaining 5 heads in 10 tosses of a fair coin is just under 8 %. We could find the remaining probabilities by calculating prob[ntotal,n] nine times:

```
prob[10, 1, 0.5, 0.5]
prob[10, 5, 0.5, 0.5]
prob[10, 10, 0.5, 0.5]

Out[89]= 0.00976563
Out[90]= 0.246094
Out[91]= 0.000976563
```

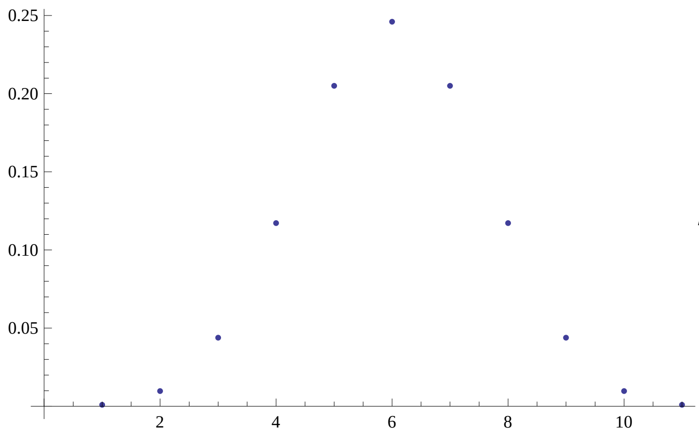
Or we could try to do all the calculations with one execution :

```
In[166]:= Clear[prob, n, ntotal]
prob[ntotal_, n_, p_, q_] := ntotal! / (n! (ntotal - n)!) p^n q^(ntotal - n)


Do[Do[If[r == 0,
   Print["ntotal = ", 10^m, "   n = 1 ", "probability = ", prob[10^m, 1, 0.5, 0.5]],
   Print["ntotal =  ", 10^m, "  ", "n = ", r 10^m / 2, "   ", "probability = ",
    prob[10^m, r 10^m / 2, 0.5, 0.5]]], {r, 0, 2}], {m, 1, 3}]


ntotal = 10   n = 1 probability = 0.00976563
ntotal =  10  n = 5   probability = 0.246094
ntotal =  10  n = 10   probability = 0.000976563
ntotal = 100   n = 1 probability = 7.88861 × 10^-29
ntotal =  100  n = 50   probability = 0.0795892
ntotal =  100  n = 100   probability = 7.88861 × 10^-31
ntotal = 1000   n = 1 probability = 9.33264 × 10^-299
ntotal =  1000  n = 500   probability = 0.025225
ntotal =  1000  n = 1000   probability = 9.33264 × 10^-302
```

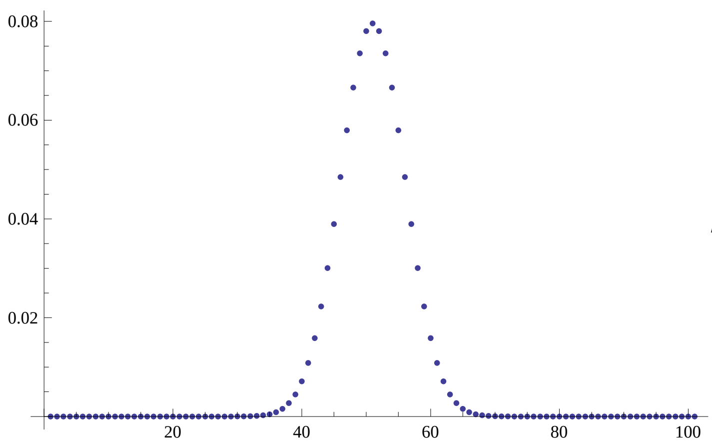
As you should have found, the probabilities of obtaining only 1 or N heads out of 100 or 1000 tosses of a coin are small. How small? Consider that the age of the universe is approximately 13.5 billion years or approximately 5 10^{17}s and that there are approximately 7.1 billion people on the planet (http://www.census.gov/popclock/). If everyone on the planet tossed 100 coins each second for the entire existence of the universe, what is the probability that just one of those events resulted in either 1 head out of 100 or all heads out of 100 tosses? This helps us to understand the role of very large numbers in thermodynamics and statistical mechanics.

b) We could write three separate commands for producing each plot, let' s see if we can do this in one command :

```
In[175]:= Table[ListPlot[Table[prob[10^m, n, 0.5, 0.5], {n, 0, 10^m}], PlotRange -> All], {m, 1, 3}]
```



,



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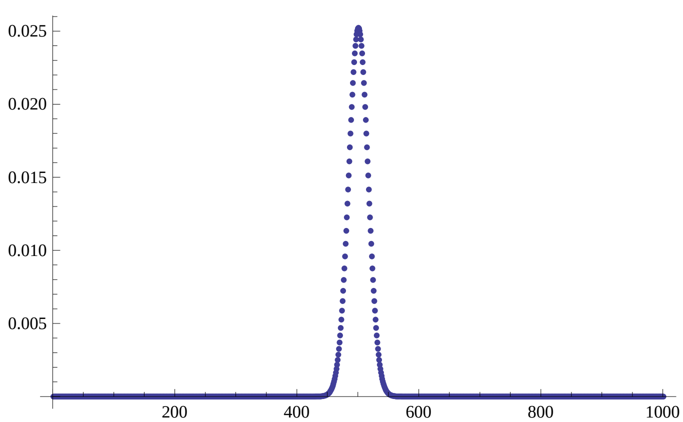
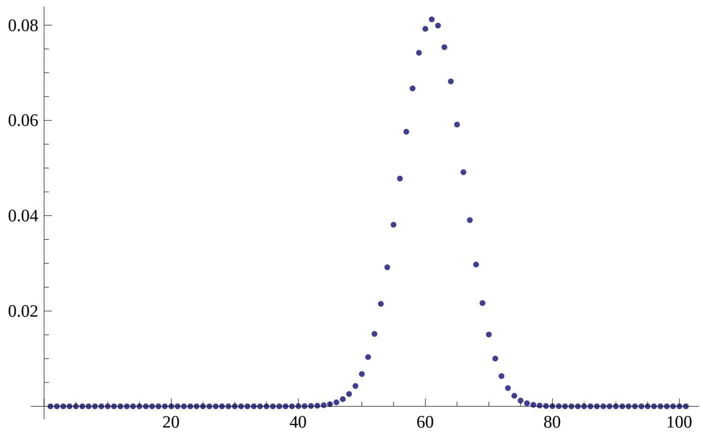


c) Comparing these graphs should reveal very quickly that the height of the most probable outcome decreases as the number of tosses increase (since there are so many more possible ways to distribute the outcomes), and that the peak becomes sharper and sharper. In class discussion and in future assignments, we will consider ways to quantify how the sharpness varies as a function of N.

d) Using our four dimensional definition of probability, it is easy to plot :

In[178]:=
```
ListPlot[Table[prob[100, n, 0.6, 0.4], {n, 0, 100}], PlotRange → All]
```

Out[178]=



The maximum of the probability distribution shifts to n = 60 as we would expect for an asymmetric coin flip where we expect to obtain a heads 60 % of the time.