

PHYS 328

HOMEWORK #3--Solutions

Due : 19 September 2013

1. Problem 1 .33, p. 23 of text.

Solution : a) The work done on the gas is negative in Step A, since the gas is expanding (and therefore the gas must do work on the outside environment). There is no work done in Step B since the process is isochoric (constant volume), thus no $P dV$ work is done. In Step C, the gas is compressed, thus work is done on the gas, and the work is positive.

b) The energy content of the gas can be written as $U = f N k T/2$. Using the ideal gas law, $PV = N k T$ allows us to write $U = (f/2) PV$. In Step A, V increases while P stays constant; the product PV increases along this step and therefore U increases in Step A. In Step B, PV increases because P increases while V stays constant; since PV increases, U increases. In Step C, both P and V decreases. Therefore the product PV decreases as does U .

c) Use the First Law of Thermodynamics to write $Q = \Delta U - W$. In Step A, ΔU is positive while Q is negative, so that Q must be positive. In other words, heat flows into the gas to make it expand in Step A. In Step B, W is zero and ΔU is positive; Q is also positive (and heat flows into the gas to make the pressure increase at constant volume). In Step C, ΔU is negative and W is positive, so Q is negative, meaning heat leaves the system.

d) For the total process, ΔU must be zero since the system returns to the same $P V$ values it had at the outset. The total work done is positive. We can determine this since the average pressure in Step C is greater than the pressure in Step A, so that $P \Delta V$ for Step C has a larger magnitude than for Step A. Since the First Law tells us that $\Delta U = Q + W$, $\Delta U = 0 \Rightarrow Q = -W$, so that Q for the entire process is negative.

We can summarize these results as :

<u>Parameter</u>	<u>Step A</u>	<u>Step B</u>	<u>Step C</u>	<u>Total Process</u>
W	< 0	0	> 0	> 0
ΔU	> 0	> 0	< 0	0
Q	> 0	> 0	< 0	< 0

2. Problem 1 . 34, p. 23 of text

Solutions : For this problem, we will make frequent use of the relations:

$$PV = N k T \text{ (Eq. A)}$$

$$U_{th} = \frac{f}{2} N k T = \frac{f}{2} PV \text{ (Eq. B)}$$

$$\Delta U = Q + W \text{ (Eq. C)}$$

$$W = - \int P dV \text{ (Eq. D)}$$

Step A : The volume is fixed and heat is added to increase the pressure. Since the volume is fixed, the work done is zero. The change in thermal energy is given by :

$$\Delta U_A = \frac{f}{2} \Delta(PV) = \frac{f}{2} V_1 (P_2 - P_1)$$

The first law of thermodynamics tells us that this is also the heat input for step A.

Step B : Here, the gas expands at constant pressure so the work done is negative (since the gas is doing work on the environment). The amount of work is :

$$W_B = - P_2 (V_2 - V_1)$$

The change in thermal energy is :

$$\Delta U_B = \frac{f}{2} \Delta(PV) = \frac{f}{2} P_2 (V_2 - V_1)$$

The first law of thermo (Eq. C) tells us that Q for Step B is :

$$Q_B = \Delta U - W = \frac{f}{2} P_2 (V_2 - V_1) + P_2 (V_2 - V_1) = \left(\frac{f+2}{2} \right) P_2 (V_2 - V_1)$$

Step C : The work is zero since ΔV is zero. Thus, $\Delta U = Q$, and we find ΔU from :

$$\Delta U_C = Q_C = \frac{f}{2} \Delta(PV) = \frac{f}{2} V_2 (P_1 - P_2) = \frac{-f}{2} V_2 (P_2 - P_1)$$

Step D : Here the pressure is held constant and positive work is done on the gas to compress it. We have that :

$$W_D = - P_1 (V_1 - V_2) = P_1 (V_2 - V_1)$$

$$\Delta U_D = \frac{f}{2} \Delta(PV) = \frac{f}{2} P_1 (V_1 - V_2) = \frac{-f}{2} P_1 (V_2 - V_1)$$

$$Q_D = \Delta U_D - W_D = \frac{-f}{2} P_1 (V_2 - V_1) - P_1 (V_2 - V_1) = - \left(\frac{f+2}{2} \right) P_1 (V_2 - V_1)$$

(Notice the sign changes; writing expressions this way will help in the next part when we sum over the entire cycle.)

For the entire cycle : To find the values of W, Q and ΔU for the entire cycle, sum the contributions from the four steps :

$$W_{\text{total}} = 0 - P_2(V_2 - V_1) + 0 + P_1(V_2 - V_1) = (P_1 - P_2)(V_2 - V_1) = - (P_2 - P_1)(V_2 - V_1)$$

$$\begin{aligned} \Delta U_{\text{total}} &= \frac{f}{2} V_1 (P_2 - P_1) + \frac{f}{2} P_2 (V_2 - V_1) - \frac{f}{2} V_2 (P_2 - P_1) - \frac{f}{2} P_1 (V_2 - V_1) = \\ &= \frac{f}{2} (P_2 - P_1)(V_1 - V_2 + V_2 - V_1) = 0 \end{aligned}$$

$$Q_{\text{total}} = -W = (P_2 - P_1)(V_2 - V_1)$$

Do these results make sense? First, we expect that ΔU for the entire cycle should be zero since we begin and end at the same values of P and V ; by the ideal gas law, T is determined by these values, so we should have the same value of T at the start and end of the cycle. Since U depends on T , no change in T means there is no change in U .

Notice that the work is just the area of the rectangle in the PV diagram. Since the expansion (negative work) occurred at a higher pressure than the compression (the positive work), the total work is negative. Since $\Delta U = 0$, $Q = -W$ and we reproduce this as we expect.

3. Problem 1 . 37, p. 26 of text.

Solution : We use the relationship :

$$V T^{f/2} = C$$

So that we can write :

$$V_1 T_1^{f/2} = V_2 T_2^{f/2}$$

and use the values :

$$T_1 = 300 \text{ K}, f = 5, \frac{V_2}{V_1} = \frac{1}{20} :$$

$$\left(\frac{T_1}{T_2}\right)^{f/2} = \frac{V_2}{V_1} \Rightarrow \frac{T_1}{T_2} = \left(\frac{1}{20}\right)^{2/f} \Rightarrow \frac{T_1}{T_2} = \left(\frac{1}{20}\right)^{2/5} = 0.3 \Rightarrow T_2 = \frac{300 \text{ K}}{0.3} = 1000 \text{ K}$$

The air is heated to a sufficiently high temperature that it ignites on its own without the need for a spark.

4. Problem 1 . 40, p. 27 of text, 10 pts each part.

Solutions : a) In this problem, we will make use of the basic relations we have learned about thermodynamics, equipartition, and ideal gases. In adiabatic expansion, $Q = 0$ so we can write :

$$\Delta U = W$$

In this case, the work is due to expansion, so we have :

$$dU = -P dV$$

Appealing to the equipartition theorem, we write :

$$U = \frac{f}{2} N k T = \frac{f}{2} P V$$

Taking differentials :

$$dU = \frac{f}{2} N k dT = \frac{f}{2} (P dV + V dP) \Rightarrow N k dT = P dV + V dP$$

But, since we know that $dU = -P dV$:

$$N k dT = \frac{-f}{2} N k dT + V dP$$

where we have set $-P dV = (f/2) N k dT$. Collecting terms gives us :

$$N k dT \left(1 + \frac{f}{2} \right) = V dP$$

Using the ideal gas law to eliminate V from the right hand side :

$$N k dT \left(1 + \frac{f}{2} \right) = \frac{N k T dP}{P}$$

Canceling terms and rearranging :

$$\frac{dT}{dP} = \frac{2}{2+f} \frac{T}{P}$$

b) We begin by employing the chain rule :

$$\frac{dT}{dz} = \frac{dT}{dP} \cdot \frac{dP}{dz}$$

We know dT/dP from part a),

and we know dP/dz from prior work in the course :

$$P(z) = P_0 e^{-\mu m_H g z / (k T)}$$

Taking the derivative of this expression for $P(z)$ gives us :

$$\frac{dP}{dz} = -\frac{\mu m_H g}{k T} P$$

Using this expression for dP/dz and our answer in a) for dT/dP , and we find our expression for the dry adiabatic lapse rate :

$$\frac{dT}{dz} = \frac{dT}{dP} \cdot \frac{dP}{dz} = \left(\frac{2}{2+f} \frac{T}{P} \right) \left(-\frac{\mu m_H g}{k T} P \right) = \frac{-2}{2+f} \left(\frac{\mu m_H g}{k} \right)$$

Our expression for dT/dz involves only constants with the values :

$f = 5$ for a diatomic gas a room temp

$$\mu = 29$$

$$m_H = 1.67 \times 10^{-27} \text{ kg}$$

$$g = 9.8 \text{ m s}^{-2}$$

$$k = 1.38 \times 10^{-23} \text{ J/K}$$

We can calculate the value of the lapse rate :

$$\frac{dT}{dz} = \frac{-2}{7} (29 \times 1.67 \times 10^{-27} * 9.8 / 1.38 \times 10^{-23})^{\circ} \text{ K/m} = -0.0098^{\circ} \text{ K/m} \approx -10^{\circ} \text{ K/km}$$

This means that a rising parcel of air will cool at the rate of 10 C/km as long as the parcel does not condense (form clouds).

5. Problem 1 . 43, p. 31 of text.

Solution. There are 12 degrees of freedom for the water molecule; 3 translational, 3 rotational and since there are 3 modes of vibration (and each vibrational mode has a potential and a kinetic degree of freedom), there is a total of 12 degrees of freedom. Thus, if the only source of thermal energy were stored in quadratic degrees of freedom we would expect that thermal energy of a single molecule of water would be :

$$C_V = \left(\frac{\partial U}{\partial T} \right)_V = \frac{\partial}{\partial T} \left(\frac{1}{2} f k T \right) = \frac{1}{2} f k = 6 k = 8.28 \times 10^{-23} \text{ J/K/molecule}$$

However, we know that the measured heat capacity of water is 4200 J/kg/K. We know determine the specific heat for one molecule. The molecular mass of water is 18 g or 0.018 kg, so there are 1000 g/18 g/mole = 55.6 moles in 1 kg of water. We know Avogadro's number, so the heat capacity per molecule is :

$$\begin{aligned} & \text{heat capacity per molecule} = \\ & \frac{4200 \text{ J/kg/K}}{55.6 \text{ mole/kg} * 6 \times 10^{23} \text{ molecules/mole}} = 1.26 \times 10^{-22} \text{ J/kg/molecule} \end{aligned}$$

We can see that the observed value is approximately 50 % greater than the value calculated assuming only quadratic forms of energy, showing that the energy stored in hydrogen bonds is a significant effect in the thermal properties of water. More specifically, we can find how many degrees of freedom there would have to be to generate such a large heat capacity from only quadratic modes :

$$\frac{1}{2} f k = 1.26 \times 10^{-22} \text{ J/kg/molecule} \Rightarrow f = 18.$$

6. Compute the gravitational potential energy of a star of mass M and radius R . Assume the star is of constant composition and density throughout. (Hint : Consider the star to be formed by a series of concentric shells of mass dm and thickness dr that fall onto the star from infinity.)

Solution : Let's begin by considering a spherical shell of radius r and thickness dr accreting onto a pre-existing sphere of radius r . If $dm(r)$ is the mass of the spherical shell and $m(r)$ is the mass of the underlying sphere, the increment of gravitational potential between them is :

$$dV = -\frac{G m(r) dm(r)}{r}$$

The mass of the shell and sphere can be related to volume through density, ρ :

$$dm(r) = 4\pi\rho r^2 dr \text{ and } m(r) = \frac{4}{3}\pi\rho r^3$$

Thus, we can rewrite eq.(1) as :

$$dV = \frac{-16}{3} \frac{G\pi^2\rho^2 r^5 dr}{r} = \frac{-16}{3} G\pi^2\rho^2 r^4 dr$$

If this represents the increment of gravitational potential to add a spherical shell to the underlying star, the gravitational potential for the entire star is :

$$V = -\int_0^R \frac{16}{3} G\pi^2\rho^2 r^4 dr = \frac{-16}{15} G\pi^2\rho^2 R^5$$

Now, we can find an expression for ρ knowing that the mass and radius of the star are M and R :

$$\rho = \frac{M}{(4/3)\pi R^3} = \frac{3M}{4\pi R^3}$$

Substitute this expression for ρ into Eq .2 :

$$V = -\frac{-16}{15} G\pi^2\rho^2 R^5 = \frac{-16}{15} G\pi^2 \left(\frac{3M}{4\pi R^3}\right)^2 R^5 = \frac{-9}{15} \frac{GM^2}{R} = \frac{-3}{5} \frac{GM^2}{R}$$

This expression represents the gravitational energy binding a sphere. You get a very similar expression if you calculate the electrical binding energy for a sphere of charges of radius R and charge Q (with k replacing G and Q replacing M).

7. Problem 1.55, pp. 36 - 37 of the text, parts a), b), c), and e) (You have done part d) more accurately). (10 pts a), 10 pts b), 5 pts c), 10 pts e))

Solutions :

a) We consider two particles traveling in a circular orbit of radius r under their mutual gravitational attraction. Each particle will have a kinetic energy of $1/2 m v^2$, so the total KE of the two particles

is $m v^2$. The two particles are a distance of $2r$ apart, so the potential energy between them is:

$$V = - \frac{G m * m}{2 r} = - \frac{G m^2}{2 r} \quad (1)$$

To compare these expressions, we would like to have a way to express v in terms of m and r . We can do this by recognizing the force acting on each particle is the gravitational force exerted by the other, and that this force must equal the centripetal force experienced by each particle. Thus, we can write the gravitational force acting on one of the particles as :

$$F_{\text{grav}} = \frac{G m^2}{(2r)^2} \quad (2)$$

and the centripetal force is :

$$F_{\text{cent}} = \frac{m v^2}{r} \quad (3)$$

Combining Eqs. (4) and (5) :

$$\frac{m v^2}{r} = \frac{G m^2}{4 r^2} \Rightarrow v^2 = \frac{G m}{4 r} \Rightarrow \text{KE} = m v^2 = \frac{G m^2}{4 r} \quad (4)$$

and comparing the result of Eq. (6) to Eq. (3) we see that :

$$V = - 2 \text{KE} \quad (5)$$

b) The total energy of the star is just the sum of kinetic and potential, so we have :

$$U_{\text{total}} = V + \text{KE} = -2 \text{KE} + \text{KE} = - \text{KE}$$

The total energy of the star is negative, and its magnitude is equal to the total kinetic energy of the particles in the star. Therefore, adding energy to the star decreases the total kinetic energy. Since we learned earlier in the chapter that total kinetic energy is related to temperature, this means that **increasing** the total energy of a star results in **cooling** the star and increasing the gravitational binding energy. Welcome to my ~~world~~ universe.

c) We learned that heat capacity can be written as :

$$C = \frac{dU}{dT}$$

The equipartition theorem tells us that the total kinetic energy is $(3/2) N k T$, and the virial theorem shows us that the total energy of the star is equal to the negative of this quantity, therefore we have :

$$U = - KE = - \frac{3}{2} N k T \Rightarrow C = \frac{dU}{dT} = - \frac{3}{2} N k.$$

Please note that the minus sign is not a typo; the heat capacity of a star is negative. This means that as you add energy to the star, the temperature decreases, a conclusion we reached in part b).

e) We know from part a) that $KE = - PE/2$, and we have an expression for the PE from problem 6, equating KE and $- PE/2$ yields :

$$KE = - PE / 2 \Rightarrow \frac{3}{2} N k T = - \left(\frac{-3}{5} \frac{G M^2}{R} \right) / 2$$

Solving for T :

$$T = \frac{G M^2}{5 N k R}$$

Using standard values gives us :

$$G = 6.7 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1}$$

$$M_{\text{sun}} = 2 \times 10^{30} \text{ kg}$$

$$R_{\text{sun}} = 7 \times 10^8 \text{ m}$$

If the sun is composed only of protons and electrons, we can assume all the mass comes from protons so the number of protons is :

$$\# \text{ protons} = \text{mass of sun} / \text{mass of proton} = 2 \times 10^{30} \text{ kg} / 1.67 \times 10^{-27} \text{ kg} \approx 10^{57} \text{ protons}$$

Assuming the sun is completely ionized (and this is a good approximation for most of its matter), there are as many free electrons as protons, so the total number of particles in the sun is twice this number. Estimating the temperature of the sun :

$$T = \frac{(6.7 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1})(2 \times 10^{30} \text{ kg})^2}{5(2 \times 10^{57})(1.38 \times 10^{-23} \text{ J/K})(7 \times 10^8 \text{ m})} \approx 3 \times 10^6 \text{ K}$$

The core (central) temperature of the sun is approx 15 million degrees K, and the surface temperature is approximately 5700 K, so our very crude estimate is not too far from some average temperature of the sun.