PHYS 328
HOMEWORK #4
Solutions

Questions 3 and 4 require the use of *Mathematica* and may be submitted as hard copy or via email as a .nb file (be sure to use your Loyola address). Strive for efficiency and clarity in your programs. A portion of your score (20 %) will be based on the efficiency of your programs. The rest of the assignment must be submitted in hard copy at the beginning of class on Th. As always, all answers must show complete work.

For the probability questions, use the techniques and equations studied in Ch. 2 of the text.

1. The national powerball lottery uses two sets of balls. The first set consists of 59 sequentially numbered balls; the second set consists of 35 sequentially numbered balls. Assume equal probability of choosing any ball, and that all the balls are differently numbered. Five balls are chosen without replacement from the set of 59. Then one ball is chosen from the set of 35. Calculate the number of ways these six balls can be chosen (and thus your probability of winning the grand powerball prize).

Solution : The total number of ways 5 items can be drawn from 59 is :

$$\frac{59!}{5! \times 54!}$$

There are then 35 ways the next ball can be chosen, so the total number of ways of drawing balls in this fashion is :

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59! / (5! × 54!) × 35.0
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$$1.75224 \times 10^{8}$$

Or a bit over 175 million. Thus, your odds of winning the powerball are less than 1 in 175 million.

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2. Consider 100 flips of a fair coin :

Solutions :

a) How many possible microstates and macrostates are there? (10 pts)

Since each flip can result in one of two outcomes, and since there are 100 flips, the total number of outcomes (or microstates) is 2^{100} . Each macrostate represents the total number of heads obtained in the set; there can be 0 heads, 1 head, ...99 heads, 100 heads, for a total of 101 macrostates.

b) How many ways are there of obtaining 50 heads and 50 tails? (5)

We use combinatorial analysis to find the number of ways (i.e., the multiplicity) of exactly 50 heads and 50 tails :

$$\Omega(50) = \frac{100!}{50! \quad 50!} = 1.00891 \times 10^{29}$$

c) What is the probability of obtaining exactly 50 heads? Of obtaining exactly 60 heads?

The probability of obtaining exactly 50 heads is :

$$\text{prob}(50) = \frac{\Omega(50)}{\Omega(\text{all})}$$

where Ω (all) represents the total number of microstates. Thus,

$$\Omega(50) = 10^{29} \big/ 2^{100} = 0.0796$$

The probability of finding exactly 60 heads is :

$$\text{prob}(60) = \Omega(60) \big/ 2^{100} = \left(100! / (60! \quad 40!) \big/ 2^{100} = 0.0108$$

d) How does the second answer in part c) compare to the probability of getting 6 heads out of 10 coin flips? Why are the answers so different? (10)

The probability of getting 6 heads out of 10 is :

$$\text{prob}(6) = \Omega(6) \big/ 2^{10} = 10! / (6! / 4!) \big/ 2^{10} = 0.2051$$

These calculations show that the probability of 6 out of 10 is not the same as 60 out of 100, even though both n and N were scaled upward by a similar factor. The answer is simply that there are so many more ways we can arrange coin tosses when N is larger (i.e., the multiplicity of each macrostate is much larger) , so that the probability of any one macrostate is smaller.

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3. Write a short Mathematica program that will calculate the probability of obtaining n heads out of N flips of a fair coin. Plot the probability of obtaining n heads as a function of n for N=10, 100, and 1000. Make sure all data points are visible. (10)

Solution : I will write a program that defines the total number of microstates and the probability of any macrostate in terms of the total number of coins (a parameter I will call ntotal).

```
Clear[prob, totalstates, ntotal]
totalstates[ntotal_] := 2^ntotal
prob[n_, ntotal_] := ntotal! / (n! (ntotal - n)!) / totalstates[ntotal]
Table[Plot[prob[n, 10^m], {n, 0, 10^m}, PlotRange → All], {m, 3}]
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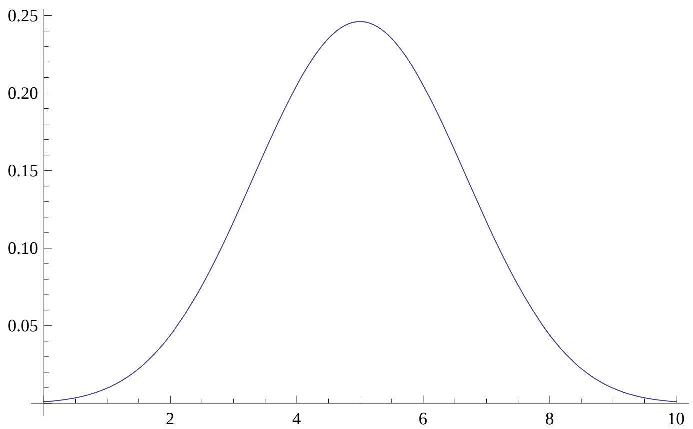

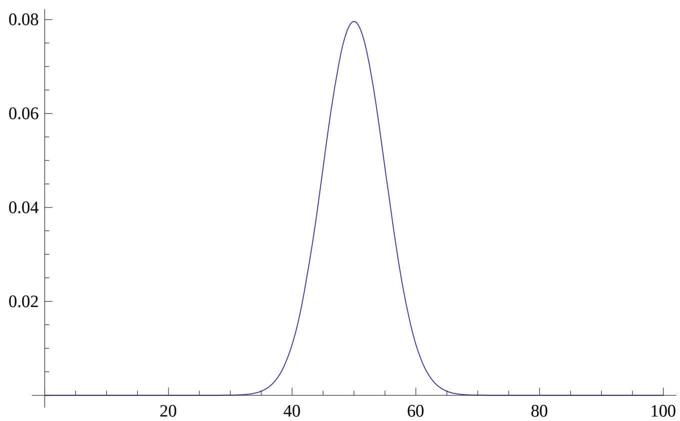

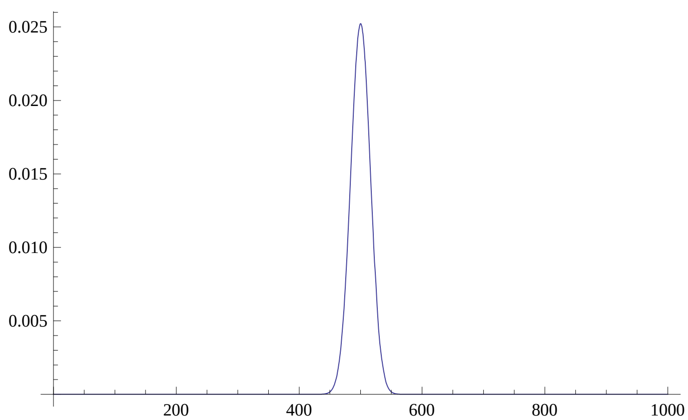

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4. Amend your program to calculate the value of n such that the probability of obtaining n heads is equal to 1/e of the maximum probability. (You may assume without proof that the maximum probability occurs when n = N/2). Calculate the "efolding value of n" for cases where N = 10, 100, 1000, 10000. Can you find a simple relationship for the spread between N/2 and n (efolding) as a function of N? (i.e., something like n (efolding) = N/100; this is not the correct relationship, but shows the format I am requesting). (20)

Solution : I write a pretty bare - bones program that calculates the probability of obtaining n heads. The next line of code finds the value of the maximum probability for a given value of ntotal. The critical line of code defines the variable "efolding"; this computes the value of n for which the probability[n] is 1/e times the maximum probability for a given ntotal. If we set ntotal = 10 we get :

```
In[365]:= Clear[prob, ntotal]
ntotal = 10;
totalstates = 2^ntotal;
prob[n_] := prob[n] = ntotal! / (n! (ntotal - n)!) / totalstates
maxprob = maxprob = prob[ntotal / 2.];
efoldingvalue =
  Catch[Do[If[prob[n] > maxprob / Exp[1], Throw[{n, prob[n] // N}]], {n, 0, ntotal}]];
Print["For ntotal = ", ntotal, ", the efolding value occurs at n = ", efoldingvalue[[1]],
 ".  The probability at n  = ", efoldingvalue[[1]], "  is ", efoldingvalue[[2]]]

For ntotal = 10, the efolding value occurs at n =
 3.  The probability at n  = 3  is 0.117188
```

Without repeating the code, I will show the output for cases where ntotal = 100, 1000, and 10,000.

For ntotal = 100, the efolding value occurs at n = 43.
For ntotal = 1000, the efolding value occurs at n = 478
For ntotal = 10000, the efolding value occurs when n = 4930.

Can we find a pattern for these results? Let' s construct a table :

ntotal	efolding value of n	spread between n/2 and n (efolding)
10	3	2
100	43	7
1000	478	22
10 000	4930	70

We can see that if ntotal increases by a factor of 10, the difference (or what I call spread or half - width) of the distance between the maximum probability and the efolding value of n does not increase by a factor of 10. Let’s consider the ratios of the “spread” to ntotal, we get:

ntotal	**spread / ntotal**
10	**0.2**
100	**0.07**
1000	**0.022**
10 000	**0.007**

Notice that the ratio of spread/ntotal decreases by a factor of 10 when the value of ntotal increases by a factor of 100; this lets us conclude that the "sharpness" of the probability distribution goes as this ratio of spread/ntotal, and that varies as $\sqrt{1/\text{ntotal}}$.

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5. Problem 2.8 in the text, all parts. (All parts 5 points).

Solutions : a) There are 21 total macrostates available to the system; they can be described as the state with 0 energy in A, 1 unit of energy in A, ... to all 20 units of energy in A.

b) The total number of microstates is given by :

$$\Omega(N, q) = \frac{(q + N - 1)!}{q!\,(N - 1)!}$$

In this case, there are 20 total oscillators and 20 units of energy, so N = q = 20 and we have :

$$\Omega(20, 20) = \frac{39!}{20!\quad 19!} = 6.89 \times 10^{10}$$

c) To find the probability of all the energy in A, we first determine the multiplicty of the state in which all energy is in A, and no energy is in B, in other words :

$$\Omega_{\text{total}}(\text{all energy in A}) = \Omega_A(20)\,\Omega_B(0)$$

There is only 1 way there can be no energy in B (all 20 oscillators have no energy), and the multiplicity of finding 20 units of energy in A is :

$$\Omega_A(\text{20 units of energy in the 10 oscillators in A}) = \frac{(20 + 10 - 1)!}{20!\quad 9!} = 1.0 \times 10^7$$

The probability of finding the entire system in this macrostate is then :

$$\text{probability of 20 units of energy in A} = \frac{10^7}{\text{total microstates}} = \frac{10^7}{6.89 \times 10^{10}} = 1.45 \times 10^{-4}$$

d) Since both solids have 10 oscillators, we expect the most likely macrostate is the one in which they each have 10 units of energy. The probability of finding the system in this macrostate is :

$$\text{prob} = \frac{\Omega_A(10\text{ units})\cdot\Omega_B(10\text{ units})}{\text{total microstates}} = \frac{\frac{(10+10-1)!}{10!\quad 9!}\cdot\frac{(10+10-1)!}{10!\quad 9!}}{6.89 \times 10^{10}} = 0.124$$

This is the probability of finding the system in the expected equilibrium state.

e) Even in our small sample (20 total oscillators), we see that the probability of being in the most likely state is 1000 times greater than the probability of being in the least likely state. Thus, if we started out with all the energy in A (or in B), and placed the two solids in thermal contact, we would expect that over time, the most likely state, the equilibrium state, is the one that would result. If we started out in the equilibrium state, we would not expect the system to evolve to a state with all the energy in A or in B. (Now, these are small samples and the probabilities are not overwhelmingly different, but as we have seen, as N increases, the equilibrium state becomes much more likely than any other state, and we expect these systems to exhibit irreversible behavior; i.e., start the system at equilibrium and it will not trend away from equilibrium).