PHYS 328
HOMEWORK #6-- SOLUTIONS

1. We can write the net vector displacement of the particle as :

$$\mathbf{D} = \mathbf{d_1} + \mathbf{d_2} + \mathbf{d_3} + \ldots \mathbf{d_N}$$

where each vector **d** has scalar length L. We can express the square of the total scalar displacement of D by taking the dot product :

$$D^2 = \mathbf{D} \cdot \mathbf{D} = (\mathbf{d_1} + \mathbf{d_2} + \mathbf{d_3} + \ldots \mathbf{d_N}) \cdot (\mathbf{d_1} + \mathbf{d_2} + \mathbf{d_3} + \ldots \mathbf{d_N}) =$$

$$(\mathbf{d_1} \cdot \mathbf{d_1} + d_1 \cdot \mathbf{d_2} + \mathbf{d_1} \cdot \mathbf{d_3} + \ldots \mathbf{d_1} \cdot \mathbf{d_N}) + (\mathbf{d_2} \cdot \mathbf{d_1} + \mathbf{d_2} \cdot \mathbf{d_2} + \mathbf{d_2} \cdot \mathbf{d_3} + \ldots \mathbf{d_2} \cdot \mathbf{d_N}) + \ldots$$

$$= d_1^2 + d_1\, d_2 \cos\theta_{12} + d_1\, d_3 \cos\theta_{13} + \ldots d_1\, d_N \cos\theta_{1\,N} + d_2\, d_1 \cos\theta_{12} + d_2^2 + d_2\, d_3 \cos\theta_{23} + \ldots$$

$$= d_1^2 + d_2^2 + d_3^2 + \ldots d_N^2 + \left(\text{a sum of terms involving } \cos\theta_{ij} \text{ where } i \neq j\right)$$

Each of the d^2 terms has a magnitude of L^2. If N is sufficiently large, then we expect the cos θ terms to be randomly oriented and will sum to zero. Therefore, since there are N d^2 terms, we have:

$$D^2 = N\, L^2 \Rightarrow D = \sqrt{N}\; L$$

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2. Begin by writing the expression for the multiplicity of an ideal gas :

$$\Omega = \frac{1}{N!} \frac{V^N}{h^{3N}} \frac{\pi^{3N/2}}{(3N/2)!} \left(\sqrt{2\, m\, U}\right)^{3N}$$

We know that S = k ln Ω, so we know we will be taking the ln of this expression. Since we know the form of the answer we are trying to prove, we will write ln Ω as :

$$\ln\Omega = N \ln V + N \ln \pi^{3/2} + N \ln (2\, m\, U)^{3/2} - \ln N! - N \ln h^3 - \ln (3N/2)!$$

Apply Stirling' s Approximation to the two terms involving factorials :

$$\ln\Omega =$$

$$N \ln V + N \ln \pi^{3/2} + N \ln (2\, m\, U)^{3/2} - (N \ln N - N) - N \ln h^3 - ((3N/2) \ln (3N/2) - 3N/2)$$

Pull out a factor of N :

$$\ln\Omega = N\left[\ln V + \ln \pi^{3/2} + \ln (2\, m\, U)^{3/2} - \ln N + 1 - \ln h^3 - (3/2) \ln (3N/2) + 3/2\right]$$

We see that there are many terms involving a power of 3/2; use the properties of ln and combine terms :

$$\ln\Omega = N\left[\ln\left(\frac{V}{N}\left(\frac{2\pi\, m\, U}{(3N/2)\, h^2}\right)^{3/2}\right) + \frac{5}{2}\right] = N\left[\ln\left(\frac{V}{N}\left(\frac{4\pi\, m\, U}{3N h^2}\right)^{3/2}\right) + \frac{5}{2}\right]$$

multiply each side by k and you have the Sackur - Tetrode equation for the entropy of an ideal gas.

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3. You have 52 choices for the first card, 51 for the second card, ..., so that the total number of ways of arranging a deck of cards is 52·51·50 ... 2·1 = 52! The total entropy is S = k ln Ω = k ln 52!

which is :

In[17]:=
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1.38 10^-23 Log[52!]
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2.1577795409824874`*^-21 J
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Omitting the factor of k, the entropy as a pure number is just ln (52!) which is :

In[19]:=
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Log[52!] // N
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Out[19]=
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156.361
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If we have an Avogadro number of particles in a card, and the card at room temperature has an energy of approximately 1/40 eV, then the total energy a single card is roughly

$$\left(\frac{1}{40}\ \mathrm{eV/particle}\right)\left(1.6\times 10^{-19}\ \mathrm{J/eV}\right)\left(10^{23}\ \mathrm{particles}\right) = 400\ \mathrm{J}$$

this answer is huge compared to the value computed above.

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4. The most likely macrostate is the one in which 3/5 of the energy is in solid A (since solid A has 3/5 of the particles), and 2/5 of the energy is in B. The multiplicity of this macrostate is the product of the multiplicities for finding A and B in this configuration. Using the equations for finding the multiplicity of an Einstein solid we get :

$$\Omega\,(\text{most likely}) = \Omega_A\,\Omega_B = \frac{(60+300-1)!}{60!\,(300-1)!}\cdot\frac{(40+200-1)!}{40!\,(200-1)!} = 6.9\times 10^{114}$$

The entropy (as a pure number) is the ln of this, or 264.4

The least likely state occurs when there is no energy in A; since there is only one way to arrange no energy in A, the multiplicity of A in this case is 1, and we have :

$$\Omega\,(\text{least likely}) = 1\cdot\Omega_B = \frac{(100+200-1)!}{100!\,(200-1)!} = 2.77\times 10^{81}$$

and the ln of this is 187.6, which is the entropy of this state as a pure number. We can find the total number of states accessible by thinking of this system as one combined Einstein solid of 500 particles and 100 units of energy. The multiplicity of this state is :

$$\Omega\,(\text{combined state}) = \frac{(100+500-1)!}{100!\,(500-1)!} = 9.3\times 10^{115}$$

The ln of this is the total entropy over long time scales. These multiplicites are huge numbers, much much greater than the lifetime of the universe in attoseconds, indicating that even in a system of a few particles, only a tiny fraction of the possible states will be accessible.

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5. In a quasistatic isothermal expansion, the temperature is constant (by the definition of isothermal), so we know that the change in thermal energy is zero. By the first law of thermodynamics :

$$\Delta U = Q + W = 0 \Rightarrow Q = -W$$

For a quasistatic process, we can compute the work done by the gas :

$$W = -\int_{V_i}^{V_f} P\,dV = -\int_{V_i}^{V_f}\frac{N k T}{V}\,dV = -N k T \ln\left(\frac{V_f}{V_i}\right) \Rightarrow Q = N k T \ln\left(\frac{V_f}{V_i}\right)$$

Dividing by T gives us trivially :

$$\frac{Q}{T} = N k \ln\left(\frac{V_f}{V_i}\right)$$

which is the expression for ΔS given by eq. 2.51 in the text.

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6. The recipe here is to find (or use) the expression for multiplicity to find the entropy, then take the partial of S with respect to U to get an expression in terms of 1/T. For an Einstein solid in the low temperature limit, we found that :

$$\Omega(N, q) = (e N / q)^q$$

Therefore, the entropy is :

$$S = k \ln \Omega = k \ln (e N / q)^q = k q \ln (e N / q) = k q[1 + \ln (N / q))$$

The total thermal energy is related to the number of energy units via :

$$U = \epsilon q$$

where ϵ is the amount of energy per energy unit. The expression for entropy becomes :

$$S = \frac{k}{\epsilon} U [1 + \ln (N \epsilon / U)]$$

Using our definition of temperature :

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k}{\epsilon}[1 + \ln (N \epsilon / U)] + \frac{k}{\epsilon} U\left[\left(\frac{N \epsilon}{U}\right)^{-1} \left(-\frac{N \epsilon}{U^2}\right)\right]$$

$$\frac{1}{T} = \frac{k}{\epsilon}[1 + \ln (N \epsilon / U)] - \frac{k}{\epsilon} = (k / \epsilon) \ln (N \epsilon / U)$$

Solving for U gives us :

$$U = N \epsilon e^{-\epsilon / k T}$$