PHYS 328
HOMEWORK #7
Solutions

The Mathematica assignments may be submitted via email as .nb files or as part of the homework you submit in class on 31 October. The rest of the assignment must be submitted in hard copy in class.

1. Problem 3.25, p. 108, parts a) - d) only.

Solutions : a) We use $S = k \ln \Omega$:

$$\Omega = \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N \Rightarrow$$

$$S = k \ln \Omega = k \ln\left[q \ln\left(\frac{q+N}{q}\right) + N \ln\left(\frac{q+N}{N}\right)\right] = k\left[q \ln\left(1 + \frac{N}{q}\right) + N \ln\left(1 + \frac{q}{N}\right)\right]$$

Any omitted terms were of the order of $\sqrt{N}$ or $\sqrt{q}$ whose ln are small compared to N or q.

b) $\dfrac{1}{T} = \dfrac{\partial S}{\partial U} = \dfrac{\partial q}{\partial U} \dfrac{\partial S}{\partial q}$

where we use the chain rule with $U = q\,\epsilon$ to make our lives a little easier. Then,

$$\frac{1}{T} = \frac{1}{\epsilon}\frac{\partial S}{\partial q} = \frac{k}{\epsilon}\left[q.\frac{1}{(q+N)/q} \cdot \frac{-N}{q^2} + \ln\left(\frac{q+N}{q}\right) + \frac{N}{\frac{q+N}{N}} \cdot \frac{1}{N}\right] = \cdot \frac{k}{\epsilon}\left[\ln\left(\frac{q+N}{q}\right)\right]$$

$$\Rightarrow T = \frac{\epsilon}{k \ln\left(\frac{q+N}{q}\right)} = \frac{\epsilon}{k \ln(1 + N/q)} = \frac{\epsilon}{k \ln(1 + N\epsilon/U)}$$

c) Starting with the expression for T :

$$\ln(1 + N\epsilon/U) = \epsilon/k\,T \Rightarrow 1 + N\epsilon/U = e^{\epsilon/k\,T} \Rightarrow U = \frac{N\epsilon}{e^{\epsilon/k\,T} - 1}$$

$$C = \frac{\partial U}{\partial T} = -\frac{N\epsilon}{\left(e^{\epsilon/k\,T} - 1\right)^2} \cdot \frac{\partial}{\partial T}\left(e^{\epsilon/k\,T} - 1\right) = -\frac{N\epsilon}{\left(e^{\epsilon/k\,T} - 1\right)^2} \cdot \left(\frac{-\epsilon}{k} \cdot \frac{1}{T^2}\, e^{\epsilon/k\,T}\right) = \frac{N\epsilon^2}{k\,T^2} \frac{e^{\epsilon/k\,T}}{\left(e^{\epsilon/k\,T} - 1\right)^2}$$

d) As $T \to \infty$, the argument of the exponential terms become small and we can use the Taylor expansion :

$$e^{x} \approx 1 + x$$

The expression then becomes :

$$\frac{N\epsilon^2}{k\,T^2}\,\frac{e^{\epsilon/k\,T}}{\left(e^{\epsilon/k\,T}-1\right)^2} \;=\; \frac{N\epsilon^2}{k\,T^2}\left(1\Big/(1+\,\epsilon/k\,T\,-\,1)^2\right) \;=\; \frac{N\epsilon^2}{k\,T^2}\cdot\frac{1}{(\epsilon/k\,T)^2} \;=\; N\,k$$

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2. Problem 3.36, p. 119. Use the results of the first problem.

Solution: a) The chemical potential is given by :

$$\mu \;=\; -\,T\,\frac{\partial S}{\partial N}$$

We use the expression for S in the first problem to find :

$$\frac{\partial S}{\partial N} = k\left[\frac{q}{1+N/q}\cdot\frac{1}{q} + \ln\left(1+\frac{q}{N}\right) + \frac{N}{1+q/N}\cdot\left(\frac{-q}{N^2}\right)\right] = k\left[\frac{q}{q+N} + \ln\left(1+\frac{q}{N}\right) - \frac{q}{q+N}\right]$$

$$\frac{\partial S}{\partial N} = k\ln\left(1+\frac{q}{N}\right) \Rightarrow \;\mu \;=\; -\,k\,T\ln\left(1+\frac{q}{N}\right)$$

b) When N >> q, q/N is a small number and ∂S/∂N varies as (k q/N) which is a small number when N is much larger than q.(Where we have used the Taylor expansion for ln(1+x) when x is small). When q>>N, ln(1+q/N) ≈ ln(q/N) which will be a number larger than 1.

We can think about these results by first imagining a system in which there are 100 particles but just 1 unit of energy. There are exactly 100 ways to distribute this unit of energy among 100 particles. If I hold q constant and increase N to 101, there are now 101 ways of distributing the one unit of energy. The multiplicity of the system increases (and therefore so does the entropy), but not by very much.

Now, consider a system of 100 units of energy and 1 particle. Now, there is only 1 way to distribute the energy (all 100 units into 1 particle). Now, hold q constant and increase N by 1. When there are 2 particles, there are now 101 ways of distributing the energy between the two particles. When q>>N, increasing N by 1 results in a much greater increase in multiplicity and therefore entropy, consistent with our earlier calcualtions for ∂S/∂N.

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3. Problem 6.3, p. 225 (use Mathematica to do the calculations and draw the graphs; submit your code with your answer either on line or in class).

Solution : The partition function is the sum of the Boltzmann factors. Since there are only two states, the partition function is :

$$Z \;=\; e^{-E_1/k\,T} + e^{-E_2/k\,T}$$

where E_1 and E_2 are the energies of the two states, and are respectively 0 eV and 2 eV. Converting eV into Joules, we get:

$$Z = e^{-0} + e^{-3.2\,10^{-19}\,\mathrm{J}/(1.38\times 10^{-23}\,\mathrm{J/K}\ T)}$$

We can write the partition function Z (T) as a Mathematica function :

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In[10]:= Clear[Z, T, k, E1, E2, eV]
eV = 1.6 × 10^-19; E1 = 0; E2 = 2 eV; k = 1.38 × 10^-23;
Z[T_] := Exp[-E1 / (k T)] + Exp[-E2 / (k T)]
Plot[Z[T], {T, 300, 300 000}]
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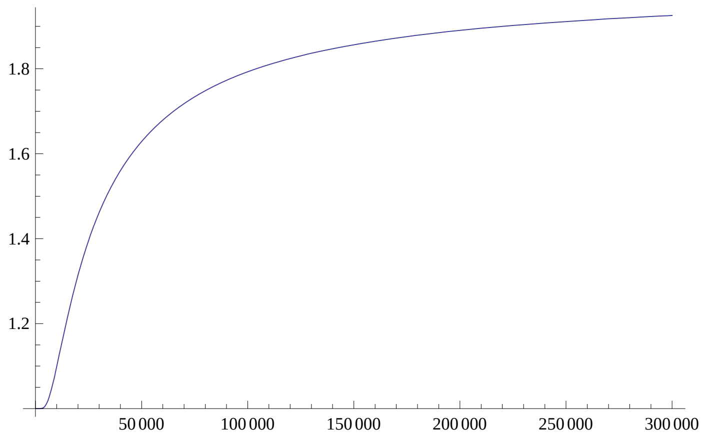
Out[13]=



In the program above, E1 and E2 are the energies of the states, eV is the conversion factor from eV to Joules, and k is Boltzmann' s constant. I can compute the values of the partition functions at the requested temperatures via :

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In[14]:= Z[T] /. T → {300, 3000, 30 000, 300 000}

Out[14]= {1., 1.00044, 1.46165, 1.92562}
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4. Problem 6.5, p. 225 parts a) and b) only.

Solution : We compute the partition function :

$$Z = e^{-(-0.05\,\mathrm{eV})/300\,k} + e^{-0} + e^{-0.05\,\mathrm{eV}/300\,k} = e^{+1.93} + 1 + e^{-1.93} = 8.05$$

The probability of each state is then the ratio of the state' s Boltzmann factor to the partition function, or :

$$P(-0.05) = \frac{e^{1.93}}{8.05} = 0.86$$

$$P(0) = \frac{1}{8.05} = 0.12$$

$$P(0.05) = \frac{e^{-1.93}}{8.05} = 0.02$$

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5. Problem 6.9, p. 227 part a) only.

Solution: If we set the ground state at zero energy, then the first excited state is 10.2 eV above the ground and the second excited state is 12.1 eV above. Remember that there is 1 ground state, 4 states corresponding to n = 2 and 9 states corresponding to n = 3. The partition function is :

$$Z = 1\,e^{-0} + 4\,e^{-10.2\text{ eV}/5800\text{ k}} + 9\,e^{-12.1\text{ eV}/5800\text{ k}} = 1 + 5.6\times10^{-9} + 2.8\times10^{-10} \approx 1$$

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6. Problem 6.13, p. 228.

Solution : In this case, our system can exist in one of two states, the proton state and the neutron state which lies at a slightly higher energy than the proton (by an amount equal to $\Delta m\, c^2$). If we set the proton's energy to zero, we can write the partition function as:

$$Z = e^{-0} + e^{-\left(\left(2.3\times10^{-30}\text{ kg}*9\times10^{16}\text{ m}^2/\text{s}^2\right)/\left(10^{11}\text{ K}*1.38\times10^{-23}\text{ J/K}\right)\right)} = 1 + e^{-0.15} = 1.86$$

Therefore, the fraction of protons is :

$$\text{fraction protons} = \frac{e^{-0}}{Z} = \frac{1}{1.86} = 0.537$$

We expect that the fraction of neutrons will be 1 - 0.537, and we verify :

$$\text{fraction neutrons} = \frac{e^{-0.15}}{Z} = \frac{0.86}{1.86} = 0.462$$