PHYS 328
HOMEWORK #8
Solutions

1. Problem 6.12, text.

Solution : As we discussed after the last group activity, this question is as much a reading exercise as it is a physics question. The question tells you that there are three molecules distributed among three excited states, or an average of one molecule per state. You are also told that there are ten molecules in the ground state, so that the probability of finding a molecule in the excited state is 1/10. Thus, we use Boltzmann' s factors to show :

$$\frac{P_2}{P_1} = \frac{e^{-E_2/k\,T}}{e^{-E_1/k\,T}}$$

where the subscripts 1 and 2 refer respectively to the ground and first excited states; P represents the probability of being in a state and E is the energy of each state. We are told that the first excited state is 0.00047 eV above the ground state, so we can write :

$$\frac{1}{10} = \frac{e^{-0.00047/(k\,T)}}{e^{-0}}$$

Using the value for k = 8.62 10^{-5}eV/K, we solve for T:

$$\ln(1/10) = -0.00047/k\,T$$

Solving for T gives 2.4 K. (The best current value for the microwave background radiation temperature is closer to 2.7 K)

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2. Problem 6.16, text

Solution : Starting from the definition of Z as the sum of all Boltzmann factors :

$$Z = \sum_s e^{-E_s/k\,T} = \Sigma e^{-\beta E_s}$$

$$\frac{\partial Z}{\partial \beta} = -\sum_s E_s\, e^{-\beta E_s} \Rightarrow \frac{-1}{Z}\frac{\partial Z}{\partial \beta} = -\sum_s (-1)\, E_s \frac{e^{-\beta E_s}}{Z} = +\sum_s E_s\, P(s) = \overline{E}$$

Where we recall that the probability of a state (P (s)) is the Boltzmann factor divided by Z, and that the final expression matches the equation for average energy given in eq. (6.16) in the text.

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3. Problem 6.17, all parts

Solutions : a) The average energy is 3 eV. The deviations are (from the most energetic to least energetic particles) : 4 eV, 1 eV, 1 eV, -3 eV, -3 eV.

b) Squaring these deviations yields positive values of 16, 1, 1, 9 and 9 $(\text{eV})^2$. The sum of these squares is 27 $(\text{eV})^2$ and the average of the squares of the deviations is 7.2 $(\text{eV})^2$. The standard deviation is the square root $\sqrt{7.2\,(\text{eV})^2}$ = 2.7 eV. This represents a good estimate of the average deviation of an individual value from the mean.

c) We begin by writing :

$$\sigma^2 = \overline{(\Delta E_i)^2} = \frac{1}{N}\Sigma_i\left(E_i - \overline{E}\right)^2 = \frac{1}{N}\Sigma_i\left(E_i{}^2 - 2\,E_i\,\overline{E} + \overline{E}^2\right) =$$

$$\frac{1}{N}\,\Sigma\,E_i{}^2 - \frac{2}{N}\,\Sigma_i\,E_i\,\overline{E} + \frac{1}{N}\,\Sigma_i\,\overline{E}^2$$

Now, let' s remember that the average energy is just a number, so can be treated as a constant and taken out of the summation.

Now, the first summation in the last line above is just the average of the squares of the energy; and we can pull the average energy out of the summations in the last two sums to give us :

$$\sigma^2 = \overline{E^2} - 2\,\bar{E}\,\Sigma_i\,\frac{E_i}{N} + \frac{\overline{E}^2}{N}\,\Sigma_i\,(1)$$

The middle summation directly above is just the average energy (the sum of the individual energies divided by N), and the final summation is just N (since we are summing the number 1 N times). We have then :

$$\sigma^2 = \overline{E^2} - 2\,\overline{E}\,\overline{E} + \overline{E}^2 = \overline{E^2} - 2\,\overline{E}^2 + \overline{E}^2 = \overline{E^2} - \overline{E}^2$$

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4. Problem 6.31, text

Solution : We write the partition function for this system as :

$$Z = \Sigma_q\,e^{-\beta c|q|}$$

We multiply and divide by Δq, and in the limit that Δq -> 0 we get :

$$Z = \frac{1}{\Delta q}\,\Sigma_q\,e^{-\beta c\,|q|}\,\Delta q = \frac{1}{\Delta q}\int_{-\infty}^{\infty} e^{-\beta c\,|q|}\,dq = \frac{2}{\Delta q}\int_0^{\infty} e^{-\beta c\,q}\,dq$$

The last step follows because of the symmetry of the absolute value function. Taking the integral gives us :

$$Z = \frac{2}{\beta\,c\,\Delta q} \equiv \frac{C}{\beta}$$

Now, we use :

$$\overline{E} = \frac{-1}{Z}\,\frac{\partial Z}{\partial \beta} = \frac{-\beta}{C}\left(\frac{-C}{\beta^2}\right) = \frac{1}{\beta} = k\,T$$

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5. Problem 6.35

Solution : The most probable value of the velocity in a Maxwellian distribution is found by determining the maximum of the probability distribution function :

$$D(v) = C\, v^2\, e^{-m v^2/2 k T}$$

where I group all of the constant coefficients as one constant defined as C. To find the maximum of this function, we take the derivative of D with respect to v and set that equal to zero :

$$\frac{dD(v)}{dv} = C\left(2 v\, e^{-m v^2/2 k T} - v^2 (m v / k T)\, e^{-m v^2/2 k T}\right) = 0$$

We can factor out a common factor of the exponential and get :

$$e^{-m v^2/2 k T}\, C\left(2 v - m v^3 / k T\right) = 0 \Rightarrow 2 v = m v^3 / k T$$

The simple solution to this equation is :

$$v = \sqrt{2 k T / m}$$

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6. Problem 6.36

Solution : For a classical gas, the number of velocity states available to the gas is very large, and the differences between energy levels is very small, so we can approximate the sum as an integral :

$$\overline{v} = \Sigma\, v\, D(v) = \int_0^\infty v\, D(v)\, dv = 4\pi\left(\frac{m}{2\pi k T}\right)^{3/2} \int_0^\infty v^3\, e^{-m v^2/2 k T}\, dv$$

where D (v) is the Maxwell speed distribution derived in class. Now, we make use of the results we obtained in class. An integral of the form is easily evaluated :

$$\int_0^\infty x\, e^{-a x^2}\, dx = \frac{-1}{2 a}\, e^{-a x^2}\Big|_0^\infty = \frac{1}{2 a}$$

Now, to find the value of

$$\int_0^\infty x^3\, e^{-a x^2}\, dx$$

we differentiate our first integral with respect to the parameter a :

$$\frac{d}{da}\int_0^\infty x\, e^{-a x^2}\, dx = \frac{d}{da}\left(\frac{1}{2 a}\right)$$

$$\int_0^\infty \frac{d}{da}\left(x\, e^{-a x^2}\right) dx = \frac{-1}{2 a^2}$$

$$\int_0^\infty -x^3\, e^{-a x^2} = \frac{-1}{2 a^2} \Rightarrow \int_0^\infty x^3\, e^{-a x^2}\, dx = \frac{1}{2 a^2}$$

For our integral of interest, the constant a = m/2 k T, so we have :

$$\bar{v} = \int_0^\infty v\, D(v)\, dv \;=\; 4\pi \left(\frac{m}{2\pi k T}\right)^{3/2} \int_0^\infty v^3\, e^{-m v^2/2kT}\, dv \;=\; 4\pi \left(\frac{m}{2\pi k T}\right)^{3/2} \cdot \frac{1}{2\,(m/2kT)^2}$$

$$\bar{v} \;=\; \frac{2}{\sqrt{\pi}}\, \sqrt{2kT/m} \;=\; \sqrt{8kT/\pi m}$$