PHYS 328
SECOND HOUR EXAM

Fall 2012

This is a closed book, closed note exam. Calculators and other electronic devices are not permitted, so please turn off and put away all electronic devices (calculators, phones, ipads, Cray XC30s, etc.). There is a sheet of equations at the end of the exam. Please make use of this during the test.

Do all your writing in blue book(s) putting your name on each blue book you use. You may do questions in any order, just make it clear to me which question you are answering. All questions must be answered clearly and completely; you must show how you reach your result. The value of each question is indicated by the numbers in parentheses. You have 75 mins for this exam.

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1. Consider a system of N spin 1/2 particles in a constant magnetic field of scalar field strength B. The particles can have one of two orientations : parallel or anti - parallel to the field. The energy of a particle can either be - μ B (parallel to the field) or + μ B (anti - parallel to the field) where μ is a constant related to the magnetic moment of the particle.

a) Show that the total internal energy of the system is :

$$U = \mu B (N - 2 N_+)$$

where N_+ and N_- are the number of particles that are parallel and anti-parallel with the field, respectively. (5)

Solution : This is shown on p. 99 of the text.

b) Making appropriate use of Stirling' s Approximation, find the entropy of this system in terms of N and N_+. (10)

Solution : See text, p. 103.

c) Derive an expression for U (T) for this system. (10).

Solution : See text, p. 104 and also the solution to problem 1 of homework set 7.

d) Express the partition function for a single particle of this system in terms of hyperbolic functions. (10)

Solution : See text, p. 232.

e) Find the average energy of a particle in this system. (10)

Solution : See text, p. 232.

f) Show that the result of e) is consistent with the result of c). (5)

Solution : Since the total thermal energy of a system, U, is just N multiplied by the average energy per particle, multiply the expression in part d) by N to obtain the expression in part c).

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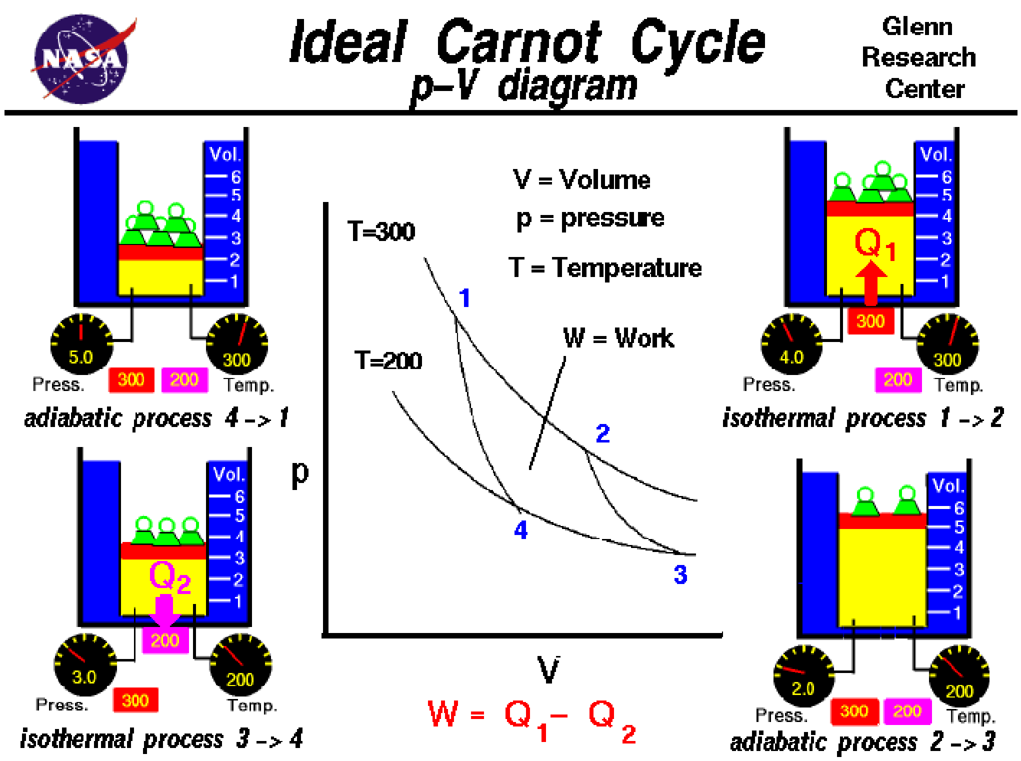
2. The Carnot cycle consists of the following four steps :

• Isothermal expansion (the power stroke) at a temperature T_h.
• Adiabatic expansion that lowers the temperature of the gas to T_C.
• Isothermal compression.
• Adiabatic compression that returns the gas to T_h.

Assume that the working fluid is an ideal gas.

a) Draw a PV diagram of this cycle, indicating on the diagram which step(s) in the cycle extract heat from the hot reservoir, and which step(s) expel heat to the cold reservoir. (5)



In this diagram, heat is extracted from the hot reservoir during the isothermal expansion (step 1 → 2) and waste heat is dumped into the cold reservoir during the isothermal compression (step 3 → 4).

Steps 2→3 and 4→1 are adiabatic, thus no heat is exchanged with the reservoirs during these steps.

b) Show that the efficiency of this cycle is :

$$e = 1 - \frac{T_c}{T_h} \ (10)$$

Solution : Since steps 1 -> 2 and 3 -> 4 are isothermal, there is no change in temperature along those curves, and therefore no change in the internal energy of the gas. The first law of thermodynamics tells us then that :

$$0 = Q + W \text{ or that } W = |Q|$$

along each isothermal curve. As the diagram shows, the total work done in one cycle is the area between the curves, which is equivalent to the difference in the absolute magnitudes of the heat exchanged during each isothermal process. We can then write, starting with the definition of efficiency :

$$e = \frac{W}{Q_h} = \frac{Q_n - Q_C}{Q_h} = 1 - \frac{Q_C}{Q_h}$$

Knowing that the magnitude of heat exchanged equals the work done along each isothermal curve, we can find expressions for the work done along each path :

$$Q_h = W_{12} = -\int_1^2 P\, dV = -\int_1^2 N\, k\, T_h \frac{dV}{V} = N\, k\, T_h \ln\left(\frac{V_2}{V_1}\right)$$

Similarly :

$$Q_c = W_{34} = N\, k\, T_c \ln\left(\frac{V_3}{V_4}\right)$$

Remember that Q is always positive in our defintion of efficiency, so that we write the ln terms to be positive. Using these expressions for the heat exchanged, we can write the efficiency as :

$$e = 1 - \frac{Q_C}{Q_h} = 1 - \frac{T_c \ln\left(\frac{V_3}{V_4}\right)}{T_h \ln\left(\frac{V_2}{V_1}\right)} \qquad (1)$$

To get to our final result, we need to show that the ratio of the ln terms is 1. We can do this by appealing to the adiabatic steps and noting that for an adiabatic process :

$$V\, T^{f/2} = \text{cst} \Rightarrow V_3\, T_c^{f/2} = V_2\, T_h^{f/2} \text{ and } V_4\, T_c^{f/2} = V_1\, T_h^{f/2}$$

Dividing these two expressions gives us :

$$\frac{V_3\, T_c^{f/2}}{V_4\, T_c^{f/2}} = \frac{V_2\, T_h^{f/2}}{V_1\, T_h^{f/2}} \Rightarrow \frac{V_3}{V_4} = \frac{V_2}{V_1}$$

Substituting this result into eq. (1) above gives us the result :

$$e = 1 - \frac{T_c}{T_h}$$

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3. Starting with the definition of Helmholtz Free Energy, use the thermodynamic identity (dU = TdS - P dV + μ dN) to show that :

$$\text{a) } dF = -S\, dT - P\, dV + \mu\, dN \quad (5)$$

Solution : We first take the total derivative of the Helmholtz Free Energy :

$$F = U - T S \Rightarrow dF = dU - T\, dS - S\, dT$$

We substitute the thermodynamic identity into the expression for dF :

$$dF = (T\, dS - P\, dV + \mu\, dN) - T\, dS - S\, dT = -S\, dT - P\, dV + \mu\, dN$$

b) Use this relationship to verify that :

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} \quad \text{and } \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} \quad (5)$$

Solution : If V and N are held constant, then dV and dN are zero, leaving us in the first case with :

$$\left(\frac{\partial F}{\partial T}\right)_{V,N} = -S$$

Holding T and V constant, and differentiating dF with respect to N gives us the second expression requested.

c) Use the results of this question and earlier questions to derive an expression for the chemical potential for a system of N identical spin 1/2 particles in a magnetic field of strength B. Assume the particles interact only with the magnetic field and not with each other.(10)

Solution : In question 1, we determined that the partition function for a single particle in this system is :

$$Z_1 = 2\cosh(\beta\, \mu\, B) \equiv 2\cosh x \quad \text{where } x = \beta\, \mu\, B$$

The subscript indicates the partition function for one particle. We also know that the Helmholtz Free Energy can be written in terms of the partition function of the system :

$$F = -kT\ln Z$$

For a system of N indistinguishable, non - interacting particles, the partition function for the entire system is related to the partition function for a single particle through :

$$Z = \frac{1}{N!} Z_1{}^N$$

Thus, we can write the partition function for the system as :

$$F = -kT\ln\left[\frac{1}{N!} Z_1{}^N\right] = -kT\,[N\ln Z_1 - \ln N!] = -kT[N\ln Z_1 - (N\ln N - N)]$$

where I have made use of Stirling' s Approximation in the last step. Now, the problem above gives us an expression for chemical potential :

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -kT[\ln Z_1 - \ln N] = -kT[\ln(2\cosh(x)) - \ln N]$$

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4. Given that :

$$\int_0^\infty e^{-a x} \sin k x \, dx = \frac{k}{a^2 + k^2}$$

where a and k are constants, find the values of :

$$\int_0^\infty x\, e^{-a x} \sin k x \, dx$$

and

$$\int_0^\infty x\, e^{-a x} \cos k x \, dx$$

Solutions : We will differentiate the integral with respect to a to obtain the first result, and with respect to k to obtain the second expression :

$$\frac{d}{da}\int_0^\infty e^{-a x} \sin k x \, dx = \frac{d}{da}\left(\frac{k}{a^2 + k^2}\right)$$

Since the integral is with respect to x, we can move the derivative inside the integral and get :

$$\int_0^\infty \frac{d}{da}\left(e^{-a x} \sin k x \, dx\right) = \frac{d}{da}\left(\frac{k}{a^2 + k^2}\right)$$

Taking the derivative of the integrand and the right hand side we get :

$$\int_0^\infty - x\, e^{-a\,x} \sin k\, x\, dx = \frac{-2\,a\,k}{\left(a^2 + k^2\right)} \Rightarrow \int_0^\infty x\, e^{-a\,x} \sin k\, x\, dx = \frac{2\,a\,k}{\left(a^2 + k^2\right)}$$

To evaluate the second integral, differentiate the integral with respect to k and get :

$$\frac{d}{dk} \int_0^\infty e^{-a\,x} \sin k\, x\, dx = \frac{d}{dk}\left(\frac{k}{a^2 + k^2}\right) \Rightarrow \int_0^\infty x\, e^{-a\,x} \cos k\, x\, dx = \frac{a^2 - k^2}{\left(a^2 + k^2\right)}$$

15 pts for entire problem.

EQUATIONS, FORMULAE AND RESULTS

$$\binom{p}{q} = \frac{p!}{q!\,(p-q)!}$$

$$S = k \ln \Omega$$

$$\ln N! = N \ln N - N$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}$$

$$Z = \sum_s e^{-\beta E(s)}$$

$$\beta = \frac{1}{k T}$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N, V}$$

$$e = \frac{W}{Q_h} = 1 - \frac{Q_C}{Q_h}$$

$$\Delta U = Q + W$$

$$W = -\int P\, dV$$

$$P V = N k T$$

$$W = N k T \ln\left(\frac{V_2}{V_1}\right)$$

$$V T^{f/2} = \text{cst}$$

$$P V^{\gamma} = \text{cst}\ (\gamma = (f+2)/f)$$

$$\overline{E} = \sum_s E(s) P(s) = \sum_s E(s) \frac{N(s)}{N}$$

$$\overline{E} = \frac{-1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \ln Z}{\partial \beta}$$

$$\cosh x = \frac{e^x + e^{-x}}{2};\ \sinh x = \frac{e^x - e^{-x}}{2};\ \tanh x = \sinh x / \cosh x$$

$$F = U - T S$$

$$G = U - T S + P V$$

$$H = U + P V$$

$$F = -k\,T \ln Z$$

$$Z = Z_1\,Z_2 \ldots Z_{N-1}\,Z_N$$

$$Z = \frac{1}{N!}\,Z_1{}^N$$