DERIVING EXPRESSIONS IN THE EDDINGTON APPROXIMATION

In our study of stellar atmospheres, we found that the plane - parallel approximation led to many useful and simplifying expressions. One of the most important of these is the Eddington approximation, in which we think of the radiation field as having two components : a component outward along the z axis that we call I_{out} and a component inward that we call I_{in}. This simple notion, coupled with definitions of intensity, flux and radiation pressure, allow us to express the mean intensity, flux, and radiation pressure in terms of I_{out} and I_{in}.

We begin with the definition of mean intensity:

$$< \mathrm{I} > = \frac{1}{4\pi}\int \mathrm{I}\, d\Omega = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \mathrm{I} \sin\theta \, d\theta \, d\phi =$$

$$\frac{1}{4\pi}\left(\int_0^{2\pi}\int_0^{\pi/2} \mathrm{I}_{\text{out}} \sin\theta \, d\theta \, d\phi + \int_0^{2\pi}\int_{\pi/2}^{\pi} \mathrm{I}_{\text{in}} \sin\theta \, d\theta \, d\phi\right)$$

It is instructive to review the limits of these integrals. The outer integrals are over the azimuthal angle ϕ and run from 0 to 2 π. The inner integrals are over the polar angle, and notice that they have different limits in the two integrals above. Because we are approximating the intensity field with two streams, one incoming and one outgoing, we integrate the outgoing stream over the upper half hemisphere, which runs from $\theta = 0$ to $\theta = \pi/2$. The incoming stream is integrated over the lower hemisphere, with limits that you see above. The term $\sin\theta \, d\theta \, d\phi$ should be familiar to you as the expression for integrating over all angles in spherical polar coordinates. We can see there is no ϕ dependence of the integrals, so the outer integrals evaluate easily to 2 π. Evaluating these integrals over θ, we get :

$$\mathbf{< I > =} \quad \frac{2\pi}{4\pi}\left(\mathrm{I}_{\text{out}}\int_0^{\pi/2} \sin d\theta + \mathrm{I}_{\text{in}}\int_{\pi/2}^{\pi} \sin\theta \, d\theta\right) \mathbf{=} \ \mathbf{\frac{1}{2}\,(I_{out} + I_{in})}$$

And this is our first major result (corresponding to equation 9.46 in the text).

Next, we find an expression for F in terms of the two intensity streams. Recall that the definition of flux is :

$$F = \int I\cos\theta\, d\Omega = \int_0^{2\pi}\int_0^{\pi} I\cos\theta\sin\theta\, d\theta\, d\phi$$

Now, setting I = $I_{out} + I_{in}$ and integrating over appropriate limits, we get:

$$F = \left(\int_0^{2\pi}\int_0^{\pi/2} I_{out}\cos\theta\sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_{\pi/2}^{\pi} I_{in}\cos\theta\sin\theta\, d\theta\, d\phi\right) =$$

$$2\pi\left(\int_0^{\pi/2} I_{out}\cos\theta\sin\theta\, d\theta + \int_{\pi/2}^{\pi} I_{in}\cos\theta\sin\theta\, d\theta\right) =$$

$$2\pi\left(\frac{I_{out}}{2} - \frac{I_{in}}{2}\right) = \boldsymbol{\pi\,(I_{out} - I_{in})}$$

and this is our second major result (text Eq. 9.47).

Finally, we use the equation for radiation pressure to find an expression relating P_{rad} to I_{in} and I_{out}. If we integrate over all space, our expression for radiation pressure is:

$$P_{rad} = \frac{1}{c}\int I\cos^2\theta\, d\Omega = \frac{1}{c}\int_0^{2\pi}\int_0^{\pi} I\cos^2\theta\sin\theta\, d\theta\, d\phi$$

Now, setting I = $I_{in} + I_{out}$, our integral becomes:

$$P_{rad} = \frac{1}{c}\left(\int_0^{2\pi}\int_0^{\pi/2} I_{out}\cos^2\theta\sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_{\pi/2}^{\pi} I_{in}\cos^2\theta\sin\theta\, d\theta\, d\phi\right) =$$

$$\frac{2\pi}{c}\left(\int_0^{\pi/2} I_{out}\cos^2\theta\sin\theta\, d\theta + \int_{\pi/2}^{\pi} I_{in}\cos^2\theta\sin\theta\, d\theta\right) =$$

$$\boldsymbol{\frac{2\pi}{3c}(I_{out} + I_{in}) = \frac{4\pi}{3c} < I >}$$

for our final major result (text eq. 9.48)