PHYS 380

HOMEWORK #2

For class discussion on Sept. 13 and submission Sept. 15.

1. Make the appropriate substitutions and do the necessary algebraic manipulations to show that eq. 2.35 follows from the previous discussion in the text.

We will use the following expressions :

$$\text{total orbital energy : } E = \frac{1}{2}\mu v_p^2 - \frac{G M \mu}{r_p} \tag{1}$$

$$r_p = \frac{(\mu r_p v_p)^2 / \mu^2}{GM(1+e)} \tag{2}$$

$$v_p^2 = \frac{G M}{a}\left(\frac{1+e}{1-e}\right) \tag{3}$$

Since the perihelion distance appears on both sides of eq. (2), we can isolate the r_p term on the left and use eq. (e) to reduce eq. (2) to :

$$\frac{1}{r_p} = \frac{v_p^2}{G M (1+e)} = \frac{\frac{G M}{a}\left(\frac{1+e}{1-e}\right)}{G M (1+e)} = \frac{1}{a(1-e)} \tag{4}$$

Substituting eqs. (3) and (4) into eq. (1) :

$$E = \frac{1}{2}\frac{G M \mu}{a}\left(\frac{1+e}{1-e}\right) - \frac{G M \mu}{a(1-e)} \tag{5}$$

Factoring and subtracting fractions :

$$E = \frac{G M \mu}{a}\left[\frac{1}{2}\left(\frac{1+e}{1-e}\right) - \frac{1}{1-e}\right] = \frac{-1}{2}\,\frac{G M \mu}{a} \tag{6}$$

QED

2. Problem 2.7, p. 54, text.

In both parts of the problem, we will use the equation for escape velocity :

$$v_{esc} = \sqrt{2 G M / R}$$

In part a, the relevant mass and radius are the mass and radius for Jupiter; in the second part the relevant mass is the solar mass and the relevant distance is the orbital distance of the Earth from the sun. In part b), we are asking how much energy is required to move an object from a point on the earth' s orbit to infinity. Numerically, we find (using approximate values for the relevant astronomical quantities_:

```
In[139]:= Clear[vesc1, vesc2, newt, solarmass, jupmass, juprad, earthdist]
newt = 6.67 × 10^(-11); solarmass = 2 × 10^30;
jupmass = 2 × 10^27; juprad = 7 × 10^7; earthdist = 1.5 × 10^11;
vesc1 = Sqrt[2 newt jupmass / juprad];
vesc2 = Sqrt[2 newt solarmass / earthdist];
Print["The escape velocity from Jupiter is  ", vesc1, " m/s"]
Print["The escape velocity from the sun from the orbit of the Earth is   ", vesc2, " m/s"]
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The escape velocity from Jupiter is  61736.8 m/s

The escape velocity from the sun from the orbit of the Earth is   42174.2 m/s
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3. Problem 2.8, p. 54, text.

a) We use Kepler' s third Law :

$$M P^2 = \frac{4 \pi^2}{G} a^3$$

where M = mass of the Earth = 6×10^{24} kg

a =

distance from center of Earth = radius earth + altitude = 6.4×10^6 m + 6.1×10^5 m

Solve for P :

In[45]:=
```
a = 6.4 × 10^6 + 6.1 × 10^5; G = 6.67 × 10^(-11); mass = 6 × 10^24;
P = (2 π / Sqrt[G mass]) a^(3 / 2);
Print["The orbital period = ", P / 3600, " hrs"]
```

```
The orbital period = 1.61926 hrs
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b) We use Kepler' s Third Law again; this time we find the value of a that will give us a geosynchronous orbit, one whose period = 1 day = 86, 400 s :

In[125]:=
```
Clear[P]
P = 86 400;
a = ((G mass P^2) / (4 π^2))^(1 / 3);
Print["the semi-major axis = ", a, " m, which is ",
 a - 6.4 × 10^6, "m above the surface of the Earth"]
```

the semi-major axis = 4.22975×10^7 m, which is 3.58975×10^7m above the surface of the Earth

c) We cannot "park" a satellite and leave it over (many) nights without some maintenance or in flight orbital adjustments. The main culprits are gravitational torques from the Moon, Sun and other astronomical objects.

4. Problem 2.13 (part a only), p. 55, text.

At aphelion and perihelion,
we know that the velocity and radius vectors are perpendicular,
so we can write the angular momentum as

$$L = \mu r v$$

Since angular momentum is conserved, we can write :

$$L_a = L_p \Rightarrow \mu r_a v_{pa} = \mu r_p v_p \Rightarrow \frac{v_a}{v_p} = \frac{r_p}{r_a}$$

where the subscripts p and a refer respectively to perihelion and aphelion. Using the defining equation of an ellipse::

$$r = \frac{a\left(1 - e^2\right)}{1 + e\cos\theta}$$

and setting our axes such that $\theta = 0$ at perihelion and $\theta = \pi$ at aphelion, we get :

$$r_p = \frac{a\left(1 - e^2\right)}{1 + e} = a\,(1 - e) \quad r_a = \frac{a\left(1 - e^2\right)}{1 - e} = a\,(1 + e)$$

Substituting these expressions into the expression for the ratio of radii gives us :

$$\frac{v_p}{v_a} = \frac{1 + e}{1 - e}$$

5. Problem 2.14 (all parts), p. 55, text

a) For this part of the problem, we can use solar system units; since the mass of a comet is much, much smaller than the mass of the sun, we can use the simplest version of Kepler' s 3 rd Law :

$$P^2 = a^3$$

where P is in yrs and a in AU. Thus :

$$a = P^{2/3} = 76^{2/3} = 17.9 \text{ AU}$$

b) For part b), we convert everything to MKS and solve for the solar mass :

In[97]:=
```
Clear[solarmass, period, dist, au, newt]
period = 76 × 365.25 × 86 400;
au = 1.496 × 10^11;
dist = 17.9 au;
newt = 6.67 × 10^(-11);
solarmass = 4 π^2 dist^3 / (newt period^2);
Print["The solar mass = ", solarmass, "  kg"]
```

```
The solar mass = 1.97584 × 10^30  kg
```

c) Using the value of eccentricity given to us and the results of the previous question, we have :

$$r_a = a(1+e) = 17.9\,(1+0.967)\text{ AU} = 35.2\text{ AU}$$

$$r_p = a(1-e) = 0.59\text{ AU}$$

The aphelion distance is beyond the orbit of Neptune; the perihelion distance is well inside the Earth' s orbit. Think about the difference in solar radiation received during one orbit of the comet; these temperature differences generate interesting and observable chemical changes in the comet as it orbits the sun.

d) Use equations 2.33 and 2.34 :

$$v_a = \frac{\sqrt{G\,M\,(1+e)}}{a\,(1-e)}$$

where a = 17.9 AU (converted to MKS), e is given in the problem and M is the mass of the sun previously determined. Substituting values, we get :

$$v_p = \sqrt{\frac{6.67\times 10^{11} * 2\times 10^{30}\text{ kg}\,(1.967)}{17.9 * 1.5\text{ x}10^{11}\text{ m} * (0.033)}} = 54,\,419\text{ m/s}$$

We know from problem 4 that :

$$\frac{v_p}{v_a} = \frac{1+e}{1-e}$$

so we have :

$$v_a = v_p\left(\frac{1-e}{1+e}\right) = 913\text{ m/s}$$

Finally, this part asks for the speed when the comet is located on the semi - minor axis. Let' s make use of the geometry of an ellipse along with the conservation of energy to see how elegant an expression we can find.

First, since we already have expressions for speed and distance from the sun for the comet at perihelion, the conservation of energy allows us to compute the speed at any point along the orbit if we know its distance from the sun. Calling b the point on the semi - minor axis and p the perihelion point, we can write :

$$\frac{1}{2}\, m v_b^2 - \frac{G M m}{r_b} = \frac{1}{2}\, m v_p^2 - \frac{G M m}{r_p}$$

Refer to diagram 2.2 and eq. 2.4 in the text, and we can write the distance of the sun to point b as :

$$r_b^2 = a^2 e^2 + a^2 \left(1 - e^2\right) = a^2 \Rightarrow r_b = a \text{ (the semi} - \text{major axis)}$$

Using this value for r_b and our previous result that r_p= a(1-e), we substitute into the conservation of energy expression and obtain:

$$\frac{1}{2}\, m v_b^2 = \frac{1}{2}\, m v_p^2 + G M m \left(\frac{1}{a} - \frac{1}{a(1-e)}\right) = \frac{1}{2}\, m v_p^2 - G M m \left(\frac{e}{a(1-e)}\right)$$

or :

$$v_b^2 = v_p^2 - G M \left(\frac{e}{a(1-e)}\right) \Rightarrow v_b = 7048 \text{ m/s}$$

e) Kinetic energy goes as the square of speed, so the ratio of kinetic energies will be the square of the ratios of the speeds, or :

$$\text{KE ratio} = (54\,419/913)^2 = 3552.$$