PHYS 380

HOMEWORK #3--SOLUTIONS

For Class Discussion on Sept. 20 and submission on Sept. 22.

1. a) Show that the gravitational binding energy of a self - gravitating sphere of mass M and radius R is :

$$U = \frac{-3}{5} \frac{G M^2}{R}$$

Assume the sphere has a constant density throughout. (Hint : Imagine the sphere being constructed by bringing in matter from infinity in a series of infinitesimal spherical shells)

Imagine that the star is constructed by infall of successive spherical shells of thickness dr. Consider the spherical shell that forms around the sphere of radius r. This sphere has a mass of $\frac{4}{3} \pi r^3 \rho$ where ρ is the density of the star. Now, the mass contained in the spherical shell dm(r), will just be $4 \pi r^2 \rho$ dr, and the gravitational potential between the spherical protostar and shell is:

$$U(r) = \frac{-G\, m(r)\, dm(r)}{r} = -\frac{G \frac{4}{3} \pi r^3 \rho \cdot 4 \pi r^2 \rho\, dr}{r} = \frac{-16}{3} G \pi^2 r^4 \rho^2\, dr$$

We integrate this expression for U (r) to find the total potential binding energy in the star :

$$U = \int_0^R U(r)\, dr = \frac{-16}{15} G \pi^2 R^5 \rho^2$$

Now, we are told the density is constant, so we know that $\rho = M \big/ \left(\frac{4}{3} \pi R^3\right)$. Substitute this into our expression for U:

$$U = \frac{-16}{15} G \pi^2 R^5 \left(\frac{M^2}{\frac{16}{9} \pi^2 R^6} \right) = \frac{-3}{5} \frac{G M^2}{R}$$

For the earth, U has a value of $2 \cdot 10^{32}$J

b) Compare this binding energy to the total energy that could be released from all the nuclear weapons in the world. Assume there are approximately 10, 000 megatons (MT) of nuclear weapons remaining. (1 MT = the energy released by the explosion of 1 million tons of trinitrotoluune (TNT); you may need to so some sleuthing to find the energy of detonation of TNT).

How likely do you think we could disrupt the structure of the Earth by simultaneously detonating all nuclear weapons?

An intensive search yields the value for the detonation energy of 1M ton of TNT as approximately $4 \cdot 10^{15}$J. Thus, the total explosive power of 10^4 warheads is only ~ 10^{19} J, a nanounit of the Earth's binding energy.

c) Compare this binding energy to the kinetic energy of an incoming sphere of radius 5 km and approaching the earth at 30 km/s. Assume the object has a uniform density of $3000\,\mathrm{kg}\big/m^3$ (roughly the density of rocky matter.)

$$\mathrm{KE} = \frac{1}{2}\,\mathrm{m\,v}^2 = \frac{1}{2}\,1.6\,\mathrm{x}10^{15}\,\mathrm{kg}\,(3\,\mathrm{x}10\verb|^|4\,\mathrm{m/s})^2 = 7\,\mathrm{x}10^{23}\,\mathrm{J}$$

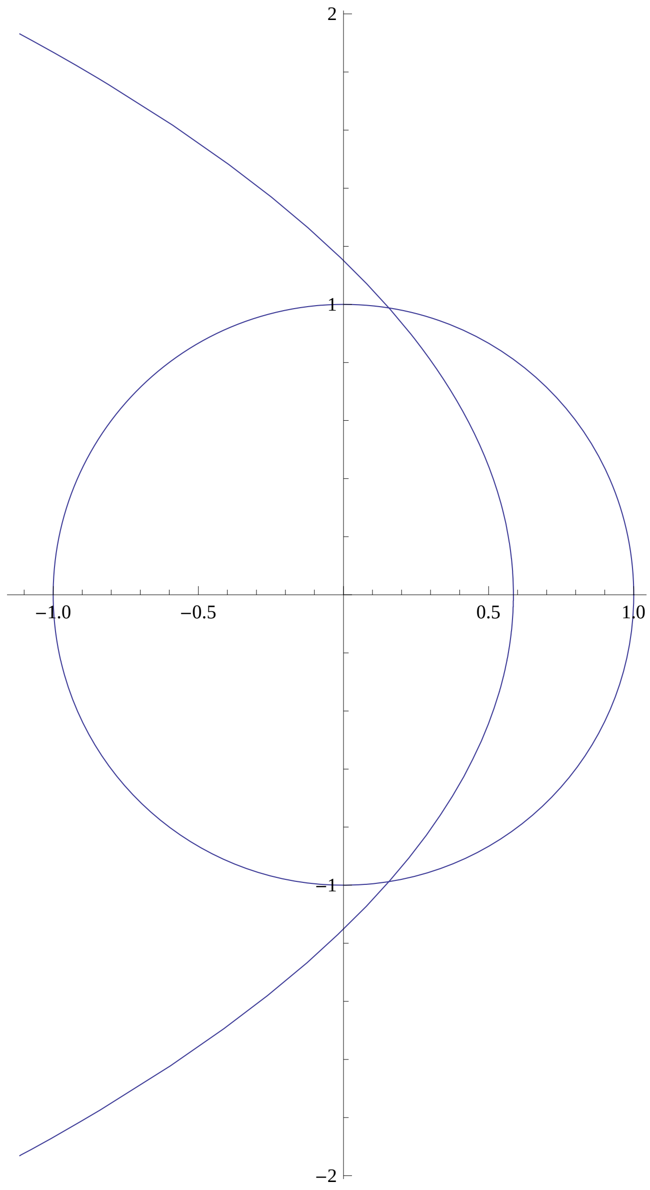
Still no fragmentation, but it will be quite a spectacle.

---

2. In the last homework assignment, you studied the orbital elements of Halley' s Comet. Use the orbital data contained in Chapter 2 to estimate how long Comet Halley is inside the Earth' s orbit. Show your work/method clearly. You might find it useful to estimate the upper bound of this time.

I will start by plotting on the same scale a portion of the orbit of Halley with the orbit of the Earth. I will assume the Earth' s orbit is circular :

```
Clear[acomet, aearth, e]
acomet = 17.9; aearth = 1; e = 0.9673;
g1 = PolarPlot[aearth, {θ, 0, 2 π}];
g2 =
  PolarPlot[ (acomet (1 - e^2) / (1 + e Cos[θ])), {θ, -2 π / 3, 2 π / 3}];
Show[g1, g2]
```



In this diagram, $\theta = 0$ occurs at cometary perihelion. The diagram also suggests that a reasonable way to start is to find the points of intersection between the Earth' s orbit and the orbit of Halley. We do so by equating their polar equations :

$$1 \;=\; 17.9\left(1-0.967^2\right)\big/(1+0.967\cos\theta) \;\Rightarrow$$

$$1+0.967\cos\theta \;=\; 1.162 \;\Rightarrow\; \cos\theta \;=\; 0.167 \;\Rightarrow\; \theta \;=\; \pm\,80.4^{\,o}$$

Halley is interior to the Earth' s orbit for 2*80.4 = 160.8 degrees of the Earth' s total path around the sun. This represents 0.45 of a year, or a time of 5.35 months. Since Halley is faster than the Earth at all points during this time, we can conclude that Halley will spend less than 5.3 months of its 76 year period interior to the Earth.

Let' s see if we can calculate a more precise number. Kepler' s second law tells us that an object orbiting the sun will sweep out equal areas in equal times. If we can compute the area swept out by Halley while it is interior to the sun, we can determine how long Halley will be interior to the Earth. Recall from Chapter 2 that the conservation of angular momentum yields :

$$\mathrm{dA} \;=\; \frac{1}{2}\,\mathrm{r}^2\,\mathrm{d}\theta$$

where dA is an element of area swept out by the comet, r is its distance from the sun, and θ is its angle with respect to the perihelion line. Then, we can calculate the fraction of the entire orbit Halley is inside the Earth' s orbit by considering the ratio :

$$\int_{-1.40}^{1.40}\frac{\mathrm{d}\theta}{(1+\mathrm{e}\cos\theta)^2}\Bigg/\int_0^{2\pi}\frac{\mathrm{d}\theta}{(1+\mathrm{e}\cos\theta)^2}$$

Notice that by taking the ratio, we can omit all constant factors such as $a\left(1-e^2\right)$. Setting e=.967 and evaluating, we find:

```
In[11]:= NIntegrate[1 / (1 + 0.967 Cos[θ]) ^2, {θ, -1.4, 1.4}] /
          NIntegrate[1 / (1 + 0.967 Cos[θ]) ^2, {θ, 0, 2 π}]

Out[11]= 0.00280835
```

And this represents the fraction of the entire orbit spent interior to the Earth' s orbit. Since the period of Halley is 12*76 months, this is equivalent to :

```
In[12]:= % 12 × 76

Out[12]= 2.56121
```

Or 2.5 months interior to the Earth, very consistent with our cruder estimate.

---

3. Starting with Eq. 3 - 22, show how you can derive eq. 3 - 24 from eq. 3 - 22; in other words, show they are equivalent expressions.

Start with :

$$B_\lambda(T)\, d\lambda = \frac{2\,h\,c^2}{\lambda^5} \frac{1}{e^{h\,c/\lambda\,k\,T} - 1}\, d\lambda \qquad (1)$$

To convert to an expression for frequency, we must make the following TWO substitutions :

$$\nu = \frac{c}{\lambda} \text{ and } d\lambda = \frac{-c}{\nu^2}\, d\nu$$

substituting these into eq. (1) gives us :

$$B_\nu(T)\, d\nu = \frac{2\,h\,c^2}{(c/\nu)^5} \frac{1}{e^{h\nu/kT} - 1} \left(\frac{c}{\nu^2}\, d\nu\right) = \frac{2\,h\,\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}\, d\nu$$

---

4. 3.2 page 81

We begin by relating flux to luminosity and distance. We know that the luminosity of the bulb is 100 W, and we are asked to find the distance where its flux will be equal to the solar constant, 1365 W/m^2.

$$F = \frac{L}{4\,\pi\,r^2} \Rightarrow r = \sqrt{\frac{L}{4\,\pi\,F}} = \sqrt{\frac{100\text{ W}}{4\,\pi * 1365\text{ W}/\text{m}^2}} = 0.076\text{ m}$$

---

5. 3.8 page 81

a) The energy emitted by a blackbody of area A and temperature T is:

$$L = A\,\sigma\,T^4$$

where σ is the Stefan - Boltmann constant. For the data provided for the human, we get :

$$L = A\,\sigma\,T^4 = 1.4\,\text{m}^2 * 5.67 \times 10^{-8}\ \text{W}\,\text{m}^{-2}\,\text{K}^{-4} * (306\,\text{K})^4 = 696\,\text{W}$$

b) Using Wien' s Law, we obtain :

$$\lambda_{\text{max}} = \frac{0.0029}{306} = 9.477 \times 10^{-6}\,\text{m} = 9477\,\text{nm}$$

far in the infrared part of the spectrum. This is why we are detectable at night with IR detecting equipment.

$$\text{c) } E = A\,\sigma\,T^4 = 1.4\,\text{m}^2 * 5.67 \times 10^{-8}\ \text{W}\,\text{m}^{-2}\,\text{K}^{-4} * (293\,\text{K})^4 = 585\,\text{W}$$

d) total energy loss = 585 W - 696 W = -111 W

---

6. 3.9 pp 81 - 82

a) $L = 4\pi R^2 \sigma T^4 = 1.17 \times 10^{31}\,\text{W} = 30\,468$ solar luminosities

b) $M = M_{\text{sun}} - 2.5 \log\left[\frac{L}{L_{\text{sun}}}\right] = 4.77 - 2.5 \log[30\,468] = -6.44$

c) $m - M = 5 \log\left(\frac{d}{10\,\text{pc}}\right) \Rightarrow m = -6.44 + 5 \log\left[\frac{123}{10}\right] = -0.99$

d) $m - M = -0.99 - (-6.44) = 5.45$

e) $F = \sigma T_e^{\,4} = 5.67 \times 10^{-8}\,\text{W}\,\text{m}^{-2}\ \text{K}^{-4} * (28\,000\,\text{K})^4 = 3.48 \times 10^{10}\ \text{W}$

f) $F_{\text{earth}} = \frac{L}{4\pi r^2} = \frac{1.17 \times 10^{31}\,\text{W}}{4\pi\left(3.79 \times 10^{18}\,\text{m}\right)^2} = 6.48 \times 10^{-8}\,\text{W}\,\text{m}^{-2}$

(Remember to convert parsecs $\rightarrow$ meters;
$1\,\text{pc} = 3.26\,\text{ly} = 3.26 * 3.15 \times 10^7\,\text{s/yr} \cdot 3 * 10^8\,\text{m/s} = 3.08 \times 10^{16}\,\text{m}$)

g) $\lambda_{max} = \dfrac{0.0029}{T} = \dfrac{0.0029}{28\,000} = 1.04 \times 10^{-7}\ m = 104\ nm$

---

7. 3.12 page 82

We want to find when the Planck function reaches a maximum. Finding the extrema of a function requires setting the first derivative to zero. Starting first with the wavelength description of Planck' s law, we have :

$$\frac{d}{d\lambda} B_\lambda(T) = \frac{d}{d\lambda}\left[\frac{2\,h\,c^2}{\lambda^5}\left(e^{h\,c/\lambda\,k\,T} - 1\right)^{-1}\right] =$$

$$2\,h\,c^2\left(\frac{-5}{\lambda^6}\left(e^{h\,c/\lambda\,k\,T} - 1\right)^{-1} + \frac{1}{\lambda^5}\cdot(-1)\left(e^{h\,c/\lambda\,k\,T} - 1\right)^{-2}\cdot\left(-\frac{h\,c}{\lambda^2\,k\,T}\right)\cdot e^{h\,c/\lambda\,k\,T}\right) = 0$$

Rearranging and dividing through by the commone exponential factor, we get :

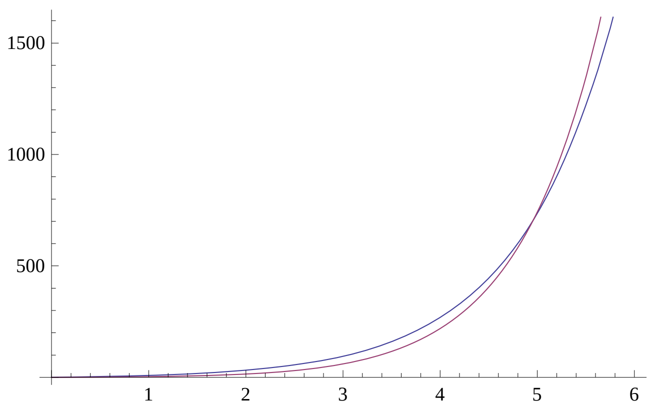
$$\frac{5}{\lambda^6} = \frac{h\,c}{kT}\cdot\frac{1}{\lambda^7}\left(e^{h\,c/\lambda\,k\,T} - 1\right)^{-1} e^{h\,c/\lambda\,k\,T} \Rightarrow 5\left(e^{h\,c/\lambda\,k\,T} - 1\right) = \frac{h\,c}{\lambda\,k\,T}\,e^{h\,c/\lambda\,k\,T}$$

Let' s set x = h c/ λ k T, so we have :

$$5\,(e^x - 1) = x\,e^x$$

Solving this equation for x will give us the conditions for a maximum in the Planck distribution. We can solve this graphically :

```
Plot[{5 (Exp[x] - 1), x Exp[x]}, {x, 0, 6}]
```



And we can see there is a solution near 5

We can use Solve :

```
Solve[5 (Exp[x] - 1) == x Exp[x], x] // N
```

InverseFunction::ifun : Inverse functions are being used. Values may be lost for multivalued inverses. ≫

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

```
{{x → 0.}, {x → 4.96511}}
```

Which Mathematica will do after reminding us this is a non - algebraic equation, or we can try :

```
FindRoot[5 (Exp[x] - 1) - x Exp[x], {x, 5}]
```

```
{x → 4.96511}
```

Which elicits much more socially acceptable behavior from Mathematica.

We now know that we obtain a maximum in the Planck function when x = 4.965. Recall that x = h c/ λ k T, and we can write that the maximum in the Planck distribution occurs when :

$$\lambda_{max} = \frac{h\,c}{4.965\,k} \cdot \frac{1}{T} = \frac{6.62 \times 10^{-34}\,\text{Js} \cdot 3\,10^{8}\,\text{m/s}}{4.965 \cdot 1.38 \times 10^{-23}\,\text{m}^2\,\text{kg}\,\text{s}^{-2}\,\text{K}^{-1}} = \frac{0.0029}{T}$$

If we use the frequency forumlation :

$$\frac{d}{d\nu} B_\nu(T) = \frac{d}{d\nu}\left(\frac{2\,h\,\nu^3}{c^2} \cdot \frac{1}{e^{h\,\nu/kT} - 1}\right) =$$

$$\frac{2\,h}{c^2}\left[3\,\nu^2 \cdot \frac{1}{e^{h\,\nu/kT} - 1} + \nu^3 + \frac{-1}{\left(e^{h\,\nu/k\,T} - 1\right)^2} \cdot e^{h\,\nu/kT} \cdot \frac{h}{kT}\right] = 0$$

Dividing through by the common denominator :

$$3 = \frac{h\,\nu}{kT}\,e^{h\,\nu/kT}\,\frac{1}{e^{h\,\nu/kT} - 1}$$

Setting x = h ν/kT, t he condition for a maximum in the Planck frequency distribution becomes :

$$3\,(e^x - 1) = x\,e^x \;\Rightarrow$$

```
FindRoot[3 (Exp[x] - 1) - x Exp[x], {x, 3}]
```

```
{x → 2.82144}
```

Finally, we can find

$$2.82 = h\nu / k T \Rightarrow v_{max} = 5.88 \times 10^{10}\ T$$