PHYS 380

HOMEWORK #6--Solutions

1. 8.6

a) We use the Boltzmann equation :

$$\frac{N_3}{N_1} = \frac{g_3}{g_1} e^{-(E_3 - E_1)/kT}$$

$$\text{where } g_3 = 18, \ g_1 = 2, \ E_3 - E_1 = \frac{8}{9} * 13.6 \text{ eV} = 12.1 \text{ eV}$$

$$\frac{N_3}{N_2} = \frac{18}{2} e^{-12.1/(8.6 \times 10^{-5} T)}$$

This ratio is 1 when :

$$1 = 9 e^{-1.41\, 10^5/T} \Rightarrow T = 64,\ 100 \text{ K}$$

b) Boltmanning again, this time with T = 85, 000 :

$$\frac{N_3}{N_1} = \frac{N_3}{N_2} = 9 e^{-1.41\, 10^5/85\,000} = 1.71$$

c) According to Boltzmann, as $T \to \infty$, the exponential factor goes to unity, so that the ratio of populations equals the ratio of statistical weights. Since the statistical weight of a level goes as $2 n^2$, this predicts that the higher energy levels are favored as $T\to\infty$. This does not happen in practice since the Saha equation shows us that atoms are completely ionized as temperatures increase.

2. 8.7

We will expand the partition function explicitly (using energy in eV and T = 10, 000 K, the term kT = 0.86) :

$$Z = \sum_{i=1}^{\infty} g_i e^{-(E_i - E_1)/k T} = g_1 e^{-(E_1 - E_1)/k T} + g_2 e^{-(E_2 - E_1)/kT} + g_3 e^{-(E_3 - E_1) kT} =$$

$$2 \; + \; 8\, e^{-10.2/0.86} + 18\, e^{-12.1/0.86} \; = \; 2 + 5.7 \times 10^{-5} \; + \; 1.4 \times 10^{-5} \; \approx \; 2$$

Extending the sum to higher energy levels will have no appreciable effect on the sum since the exponential terms will be even smaller as the energy difference increases.

3. 8.10

a) Fun with Saha. I will write a short Mathematica program to handle these calculations. Some quick notes about notation and units. Notice that I define both k and kb (Boltzmann k). "k" is measured in eV and is used in conjuction with ionization energy (also in eV); "kb" is measured in MKS and is used in all other places where the Boltzmann constant appears in Saha. All other units are in MKS. Subscripts of 2 refer to the first ionized state; subscripts of 3 refer to the second ionized state. Notice that I use electron pressure as an argument of the function call; this is to facilitate calculations in the next problem.

```
In[445]:= Clear[f2, f3, pe, k, me, temp, h, ion2, ion3, Z1, Z2, Z3]
k = 8.6 × 10^-5; h = 6.62 × 10^-34; me = 9.1 × 10^-31; Z1 = 1;
Z2 = 2; Z3 = 1; ion2 = 24.6; ion3 = 54.4; kb = 1.38 × 10^-23;
factor[temp_] := (2 π me kb temp / h^2)^(3 / 2)
f2[temp_, pe_] := (2 kb temp Z2 / (pe Z1)) factor[temp] Exp[-ion2 / (k temp)]
f3[temp_, pe_] := (2 kb temp Z3 / (pe Z2)) factor[temp] Exp[-ion3 / (k temp)]

Text[Style["Ratio            T=5000                    T=15000          T = 25000", Bold]]

Print["He+/He:   ", "   ", f2[5000, 20], "    ", f2[15000, 20], "      ", f2[25000, 20]]
Print["He++/He+: ", "   ", f3[5000, 20], "    ", f3[15000, 20], "    ", f3[25000, 20]]
```

Ratio	T=5000	T=15000	T = 25000
$\frac{He^{+}}{He}$:	1.68161×10^{-18}	0.960106	7074.55
$\frac{He^{++}}{He^{+}}$:	3.35775×10^{-49}	2.227×10^{-11}	0.00169089

As you can see, the ionization rates are very low until temperatures approach 25, 000 K.

b) Start with :

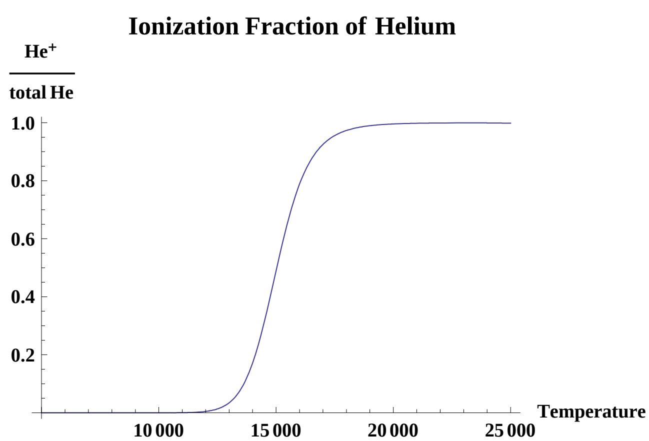
$$\frac{N_{II}}{N_{total}} =$$

$$\frac{N_{II}}{N_I + N_{II} + N_{III}} = \frac{N_{II} / N_I}{1 + N_{II} / N_I + N_{III} / N_I} = \frac{N_{II} / N_I}{1 + N_{II} / N_I + (N_{III} / N_{II}) (N_{II} / N_I)}$$

c)

In[455]:=
```
Plot[f2[temp, 20] / (1 + f2[temp, 20] + f3[temp, 20] f2[temp, 20]),
 {temp, 5000, 25 000}, PlotLabel → "     Ionization Fraction of Helium",
 LabelStyle → Directive[Bold], AxesLabel → {"Temperature", "He⁺/total He"}]
```

Out[455]=



We can see from the curve above that 50 % ionization occurs at approximately 15, 000 K.

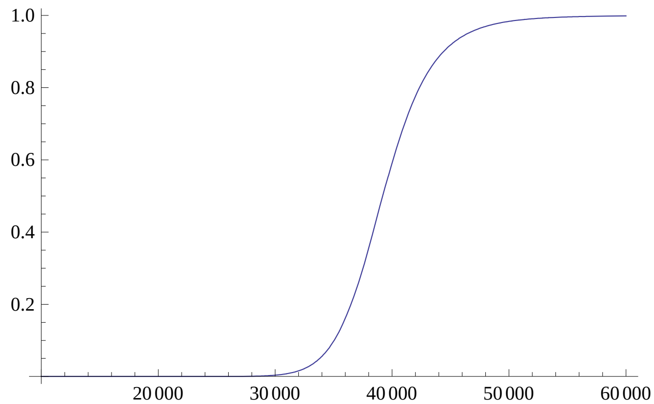
4. 8.11

We will do a very similar analysis to the previous problem, except now that the value of pe = 1000 and the ratio we are studying is :

$$\frac{N_{III}}{N_{total}} = \frac{N_{III}}{N_I + N_{II} + N_{III}} = \frac{N_{III} / N_{II}}{(N_{II} / N_I)^{-1} + 1 + N_{III} / N_{II}}$$

```
In[222]:= Plot[f3[temp, 1000] / ((1 / f2[temp, 1000]) + 1 + f3[temp, 1000]), {temp, 10 000, 60 000}]
```

Out[222]=



And the 50 % ionization zone occurs at approx 40, 000 K.

5. 8.12

By now you are old friends with Saha (remember to use the electron density formulation) :

$$f \equiv \frac{H^+}{H} = \frac{2\,Z_{II}}{n_e\,Z_I}\left(\frac{2\,\pi\,m_e\,k\,T}{h^2}\right)^{3/2} e^{-13.6\,\text{eV}/k\,T} = 2.44$$

for the input given. Recall now that :

$$\frac{H^+}{H\text{ total}} = \frac{H^+}{H + H^+} = \frac{f}{1+f} = \frac{2.44}{3.44} = 0.71$$

This version predicts that only 71 % of all H atoms are ionized, which certainly seems low for such a high temperature. The confounding factor in this case is the very high electron density; at such high densities, the electron pressure perturbs the electron orbital energies, contributing to a higher rate of ionization.

6. 8.14

The Saha equation tells us the ionization fraction of an ion, a key piece of information in predicting the strength of a spectral line at a certain set of atmospheric conditions. In order for stars to exhibit the same range of spectral lines, we expect to find atoms in the same state of ionziation; in other words, we expect the fraction f in the equation below to be roughly the same for stars of the same spectral type. We will investigate the ionization fraction for different stars of the same spectral type The key parameters here are the electron density and temperature; let' s write an abbreviated form of Saha showing the functional dependence on these factors :

$$f \equiv \frac{N_{i+1}}{N_i} = \frac{C_1}{n_e} C_2 \, T^{3/2} \, e^{-\chi/k\,T}$$

where C_1 and C_2 are constants. We know that the electron density is much lower in the giant star than the main sequence star; this will have the effect of increasing the value of f. We can investigate the temperature dependence of f by differentiating this expression with respect to T to show that f increases monotonically as temperature increases:

$$\frac{df}{dT} = \frac{C_1 \, C_2}{n_e}\left[(-1)\left(\frac{-\chi}{k\,T^2}\right)T^{3/2} + \frac{3}{2}\,T^{1/2}\right]e^{-\chi/kT} > 0$$

Therefore, since f increases as the electron density decreases the temperature must vary in a manner to offset this increase. Knowing that f varies monotonically with T, we conclude that T must decrease in order to maintain the same value of f.

7. 8.16

Fomalhaut is an A3V star (notice it is just "below" Vega and Sirius on the MS); its absolute magnitude is approximately 1.8, so knowing its visual magnitude = 1.19, we obtain :

$$d = 10^{(m-M+5)/5} = 10^{(1.19-1.8+5)/5} = 7.55\,\mathrm{pc}$$

The HR diagram is a wonderful thing.