PHYS 380
HOMEWORK #7

For discussion on Oct 27 and submission on Nov. 1

All questions from the text :

1. 9.1 This will give you a chance to do calculations with energy density, flux, and think about the nature of the radiation in your eye.

The energy density of blackbody radiation is :

$$u = \frac{4\sigma T^4}{c}$$

where σ is the Stefan - Boltzmann constant and c is the speed of light. For an eye of radius 1.5 cm, the total energy in the eyeball is

$$\text{eyeball energy} = u\,V = \frac{4\sigma T^4}{c}\left(\frac{4}{3}\pi r^3\right)$$

For T = 310 K and r = 0.0015 m, the energy in the eye cavity is 10^{-10}J

The flux from the light bulb at a distance of 1 m is :

$$F = \frac{L}{4\pi r^2} = \frac{100\text{ W}}{4\pi(1\text{ m})^2} = 7.96\text{ W}$$

The amount of energy entering the eye each second is area x flux :

$$\text{energy entering eye/sec} = 7.96\text{ W} * 10^{-5}\text{ m}^2/\text{s} = 7.96\text{ x}10^{-5}\text{ J/s.}$$

A simple dimensional analysis suggests that we should multiply J/s by time to determine the energy in the eye at a specific time; but what is the relevant time? The transit time for light to cross the eye is the diameter of the eye/speed of light, so at any given time, the amount of energy in the eye is :

$$\text{Energy in eye} = 7.96\,\text{x}10^{-5}\ \text{J/s} * \left(\frac{2 * 0.015\ \text{m}}{3\,\text{x}\,10^{8}\ \text{m/s}}\right) = 7.96 * 10^{-15}\ \text{J}$$

four orders of magnitude less than the blackbody radiation generated inside the eye. Then why is it not light when we close our eyes? Because the wavelength of maximum emission for a blackbody of 310 K is in the infrared, beyond the range of senstitivity for our eyes.

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2. 9.6

We will calculate the mean free path for nitrogen molecues at room conditions, and use this result in conjunction with the root mean square speed to determine the time between collisions. We start by computing the root mean square speed :

$$v_{rms} = \sqrt{\frac{3\,k\,T}{\mu\, m_H}} = \sqrt{\frac{3 * 1.38\ \text{x}\ 10^{-23}\ \text{J/K} * 300\ \text{k}}{28 * 1.6\ \text{x}10^{-27}\ \text{kg}}} = 526\ \text{m/s}$$

We will find the mean free path from :

$$l = \frac{1}{n\,\sigma}$$

where l is the mean free path, n is the number density of nitrogen molecules, and σ is the cross - sectional area for collisions. We will adopt the shift in reference frame used in the text where we imagine a molecule of radius 2 r is moving through an ensemble of point particles. Then, the cross sectional area for our molecule of radius 2 r becomes :

$$\sigma = \pi\,(2\,r)^2 = \pi\left(2\ \text{x}\ 10^{-10}\ \text{m}\right)^2 = 1.26\,\text{x}10^{-19}\ \text{m}^2$$

We are given the mass density of nitrogen, and can find the number density from :

$$n = \frac{\rho}{\mu\, m_H} = \frac{1.2\ \text{kg m}^{-3}}{28 * 1.6\,\text{x}10^{-27}\ \text{kg}} = 2.68\ \text{x}\ 10^{25}\ \text{m}^{-3}$$

Finally, the mean free path is :

$$l = \frac{1}{n\,\sigma} = \frac{1}{2.68\,\text{x}10^{25}\ \text{m}^{-3} * 1.26\,\text{x}10^{-19}\ \text{m}^2} = 2.96\,\text{x}10^{-7}\ \text{m}$$

(about 1000 molecular radii)

and the time between collisions is simply :

$$\text{time} = \frac{\text{mean free path}}{v_{rms}} = \frac{2.96\,\text{x}10^{-7}\,\text{m}}{526\,\text{m/s}} = 5.63\,\text{x}10^{-10}\,\text{s}$$

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3. 9.8. You should be able to obtain an expression for the optical depth of the earth' s atmosphere based in terms of the intensities I_1 and I_2 and the angles θ_1 and θ_2. You should also be able to derive an expression for the intensity at the top of the earth's atmosphere in terms of intensities and angles.

This example will show how we can use the observables, the intensities of an object measured on the surface of the earth at two different angles, to determine the intensity of the radiation field at the top of the earth' s atmosphere, and also the vertical optical depth through the atmosphere.

We know the intensity received onthe surface is

$$I_\lambda = I_{\lambda,0}\, e^{-\tau_{\lambda,0} \sec\theta}$$

where I_λ is the intensity measured on the surface of the earth, $I_{\lambda,0}$ is the intensity measured at the top of the atmosphere, θ is the angle of the object from the vertical, and $\tau_{\lambda,0}$ is the vertical optical depth. To save typing, I will delete the subscripts for λ, understanding that these are all wavelength dependent properties. Therefore, we can write I_1 and I_2 as the surface measurements when the star is observed at angles $\theta_1$and θ_2:

$$I_1 = I_0\, e^{-\tau_0 \sec\theta_1};\;\; I_2 = I_0\, e^{-\tau_0 \sec\theta_2} \tag{1}$$

Take ratios of the intensities and find :

$$\frac{I_1}{I_2} = e^{-\tau_0 (\sec\theta_1 - \sec\theta_2)}$$

Now take the natural log of both sides :

$$\ln\left(\frac{I_1}{I_2}\right) = -\tau_0 (\sec\theta_1 - \sec\theta_2)$$

Solving for vertical optical depth :

$$\tau_0 = \frac{\ln(I_1/I_2)}{\sec\theta_2 - \sec\theta_1}$$

We can now solve for vertical optical depth in equations (1) and after taking lns, obtain :

$$\tau_0 = \frac{\ln(I_1 / I_0)}{\sec\theta_1} \text{ and } \tau_0 = \frac{\ln(I_2 / I_0)}{\sec\theta_2}$$

Since τ is the same, we equate the expressions :

$$\frac{\ln(I_1 / I_0)}{\sec\theta_1} = \frac{\ln(I_2 / I_0)}{\sec\theta_2}$$

Solving for $I_{\lambda,0}$ (and after much algebra utilizing the laws for logs):

$$I_{\lambda,0} = \left(\frac{I_2^{\sec\theta_1}}{I_1^{\sec\theta_2}}\right)^{1/(\sec\theta_1 - \sec\theta_2)}$$

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4. 9.21

As indicated, start with:

$$I(0) = I_0\, e^{-\tau_{v,0}\sec\theta} - \int_{\tau_{v,0}\sec\theta}^{0} S_\lambda \sec\theta\, e^{-\tau_{v,0}\sec\theta}\, d\tau_v$$

We are told that there is no radiation entering from outside, so the first term on the right goes to zero; we are also told to assume that S_λ does not vary with position, which allows us to treat this as a constant in the integrand, making integration of the second term on the right quite simple, and we obtain:

$$I(0) = S_\lambda (1 - e^{-\tau_{\lambda,0}})$$

and we investigate the properties of this solution for the cases $\tau >> 1$ and $\tau << 1$.

If $\tau >> 1$ (optically thick case), we see that the exponential term goes to zero, leaving only $I_\lambda(0) = S_\lambda = B_\lambda$ since we know an optically thick gas will be in LTE. Thus, the intensity we observe will be the continuum blackbody radiation.

In the case of $\tau << 1$, we can EXPAND THE EXPONENTIAL IN A POWER SERIES . Truncating the series after the second term (since all higher terms will approach zero), we obtain :

$$I_\lambda(0) = S_\lambda \left(1 - \left(1 - \tau_{\lambda,0}\right) = S_\lambda\, \tau_{\lambda,0}\right.$$

We can investigate further the properties of the optically thin case. We know from the definition of optical depth that

$$\tau_{\lambda,0} = \kappa_\lambda \rho \, s \text{ or in this case } \kappa_\lambda \rho L \text{ since our length is } L$$

and that :

$$S_\lambda = j_\lambda / \kappa_\lambda \Rightarrow I_\lambda(0) = (j_\lambda / \kappa_\lambda)(\kappa_\lambda \rho L) = j_\lambda \rho L$$

This means that the intensity we observe will be large where j_λ is large; in other words, we see emission lines in the optically thin case at those wavelengths where j is large (i.e., where there are spectral lines).

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5. 9.24

We are given the hint to review the chapter on celestial mechanics to remind us of the properties of ellipses. The equivalent width of a spectral line is the width of a box (reaching to the continuum) whose area matches the area of the spectral line. We know from lots of sources (celestial mechanics, vector calculus) that the area of an ellipse is π a b where a and b are the semi-major and semi-minor axes respectively.

In this case, we are told we can model the spectral line as a semi-ellipse, so that we know its area is π a b/2. We are also told that there is zero flux at the center of the spectral line, so that we know the length of its equivalent "box" is equal to the semi-major axis of the ellipse, a. Therefore, the equivalent width of this line is:

$$W = \frac{\pi a b / 2}{a} = \frac{\pi b}{2}$$

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6. 9.26

This problem has been deferred until the next homework set, due Nov. 8, 2011.