PHYS 380
HOMEWORK #9

Solutions

1. 10.9

The peak of the Gamow spectrum occurs when the term

$$e^{-E/kT + b\,E^{-1/2}}$$

is a maximum. This means we set the derivative of this term to zero :

$$\frac{d}{dE}\left(e^{-E/k\,T + b\,E^{-1/2}}\right) = \frac{-1}{k\,T} - \frac{b}{2}\,E^{-3/2}\left(e^{-E/k\,T + b\,E^{-1/2}}\right) = 0$$

$$\Rightarrow \frac{-1}{k\,T} - \frac{b}{2}\,E^{-3/2} = 0 \Rightarrow E_0 = \left(\frac{b\,k\,T}{2}\right)^{2/3}$$

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2. 10.10

We are asked to find the ratio of energy generation rates between the pp and CNO processes; this means we compute the ratio :

$$\frac{\epsilon_{pp}}{\epsilon_{CNO}}$$

To do so, we use the expressions for these energy generation rates found in the text (eqs. 10.47 and 10.59) :

$$\epsilon_{pp} = \epsilon'_{0,pp}\,\rho\,X^2\,f_{pp}\,\psi_{pp}\,C_{pp}\,T_6^4$$

$$\epsilon_{CNO} = \epsilon'_{0,CNO}\,\rho\,X\,X_{CNO}\,T_6^{19.9}$$

where $\epsilon'_{0,\text{pp}}$ and $\epsilon'_{0,\text{CNO}}$ are constants for each process, ρ is the mass density, X is the hydrogen mass fraction, X_{CNO} is the mass fraction of carbon, nitrogen and oxygen, f_{pp}, $\psi_{\text{pp},}$ and C_{pp} are constants all of order unity describing screening, branching and higher order effects for the pp chains.

Therefore, this ratio becomes:

$$\frac{\epsilon_{\text{pp}}}{\epsilon_{\text{CNO}}} = \frac{\epsilon'_{0,\text{pp}}\, X\, T_6^4}{\epsilon'_{0,\text{CNO}}\, X_{\text{CNO}}\, T_6^{19.9}}$$

Using values cited in the problem and the text :

$$\frac{\epsilon_{\text{pp}}}{\epsilon_{\text{CNO}}} = \frac{1.08 \times 10^{-12}\ \text{W m}^3\,\text{kg}^{-2} * 0.3397 * (15.69)^4}{8.24 \times 10^{-31}\,\text{W m}^3\,\text{kg}^{-2} * 0.0141 * (15.69)^{19.9}} = 3.08$$

showing that the pp chain is the dominant mode of energy generation in the sun.

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3. 11.1

We know that the luminosity of a star is related to its radius and effective temperature according to :

$$L = 4\pi r^2 \sigma T_e^4$$

If the model presented in Fig 11.1 is accurate, then this relationhship should hold at all times and for all model values of L, R and T_e. Let's compare the current values of these parameters with the corresponding values at t=0. At t=0, we see that the solar luminosity was 0.68 of the current value, therefore we expect that

$$\frac{L_0}{L_{\text{now}}} = 0.68 = \left(\frac{R_0}{R_{\text{now}}}\right)^2 \left(\frac{T_0}{T_{\text{now}}}\right)^4$$

Using the graph in Fig. 11.1, we obtain $R_0 = 0.87\, R_{\text{now}}$; T_0 = 5621K and $T_{\text{now}} = 5777\, K$. Using these data we find:

$$\left(\frac{R_0}{R_{now}}\right)^2 \left(\frac{T_0}{T_{now}}\right)^4 = (0.87)^2 \left(\frac{5621}{5777}\right)^4 = 0.68$$

so within the limits of reading the graph, we see that the luminosity relationship holds. We could repeat this process for other sets of data, and will reach the same conclusion.

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4. 11.3

The Saha Equation gives us the ratio of the $i + 1^{st}$ ionization state to the i^{th} ionization state, therefore, we take the ratio of:

$$\frac{N_H}{N_{H^-}} = \frac{2\,k\,T\,Z_H}{P_e\,Z_{H^-}} \left(\frac{2\,\pi\,m_e\,k\,T}{h^2}\right) e^{-\chi_I/k\,T}$$

with the values (all in MKS) :

$k = 1.38 \times 10^{-23}$;

$T = 5777$ K;

$P_e = 1.5$;

$Z_H = 2$; $Z_{H^-} = 1$;

$m_e = 9.11 \times 10^{-31}$;

$h = 6.63 \times 10^{-34}$;

ionization energy $= 0.75$ eV $= 1.6 \times 10^{-19}$ J;

Writing these in Mathematica friendly format :

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In[15]:= Clear[k, t, pe, me, zh, zh1, h, ion]
k = 1.38 × 10^-23; t = 5777; pe = 1.5; me = 9.11 × 10^-31;
zh = 2; zh1 = 1; h = 6.63 × 10^-34; ion = 1.6 × 10^-19;

ratio = ((2 k t zh) / (pe zh1)) (2 π me k t / h^2)^(3 / 2) Exp[-ion / (k t)]

Out[17]= 3.02216 × 10^7
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Indicating there is much more N_H than N_{H^-}. N_{H^-} becomes an important opacity source in cooler stars.

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5. 12.2

The energy emitted by the star per second, its luminosity, is

$$L_* = 4\pi R_*^2 T_*^4$$

The amount of the star' s energy incident on a point a distance d from the star is :

$$\frac{L_*}{4\pi d^2}$$

The cross sectional area of a spherical grain to the star, i.e., the area of the grain that can absorb energy is πr_g^2 where the subscripts "*" and "g" refer respectively to the star and the grain. Therefore, the total amount of energy absorbed by the grain is:

$$\text{energy absorbed} = \frac{L_*}{4\pi d^2}\left(\pi r_g^2\right)$$

The energy emitted by the grain (assuming it emits as a blackbody is)

$$4\pi r_g^2 \sigma T_g^4$$

In radiative equilibrium, the emission and absorption rates are equal, so we have

$$\frac{L_*}{4\pi d^2}\left(\pi r_g^2\right) = 4\pi r_g^2 \sigma T_g^4 \Rightarrow 4\pi R_*^2 \sigma T_*^4 \left(\frac{\pi r_g^2}{4\pi d^2}\right) = 4\pi r_g^2 \sigma T_g^4$$

Cancelling common terms :

$$T_g^4 = \frac{R_*^2 T_*^4}{4 d^2} \Rightarrow T_g = \left(\frac{R}{2\, d}\right)^{1/2} T_*$$

For an F0 MS star, T = 7300 K and $R_* = 1.4\, R_{\text{sun}}$; if d=100AU = 1.5 x 10^{13}m:

$$T_g = \left(\frac{1.4 * 7\times 10^8 \text{ m}}{2 \text{ x } 1.5\times 10^{13} \text{ m}}\right)^{1/2} (7300 \text{ K}) = 42 \text{ K}$$

12.6

"Normal" CO is much more abundant than ^{13}CO or C^{18}O; therefore lines in these isotopic forms are likely to be much less optically thick allowing us to see deeper into the clouds. Lines of "normal" CO are much more likely to be optically thick, giving us a view only of the outer layers of the cloud.

Bonus question : 12.4 (for students needing to do an additional HW problem)

Using the equation (and noting that the full width at half max is measured in km/s for this equation), we find the column density of H to be :

$$\tau_H = 5.2\times10^{-23}\frac{N_H}{T\,\Delta\nu} \Rightarrow N_H =$$

$$\frac{T\,\Delta\nu\,\tau_H}{5.2\times10^{-23}} = \left(\frac{100\,\mathrm{K} * 10\,\mathrm{km\,s^{-1}} * 0.5}{5.2\times10^{-23}}\right) = 9.6\times10^{24}\,\mathrm{m^{-2}}$$

Since the column density is just the average density x depth of the cloud, we have

$$\text{depth of the cloud} = \frac{9.6\times10^{24}\,\mathrm{m^{-2}}}{10^{7}\,\mathrm{m^{3}}} = 9.6\times10^{17}\,\mathrm{m} = 31.2\,\text{parsecs}$$